ON THE SUBGROUPS OF ORDER A POWER OF p IN THE QUATERNARY ABELIAN GROUP IN THE GALOIS FIELD OF ORDER p^n*

BY

LEONARD EUGENE DICKSON

1. The problem of the p-section of the periods of hyperelliptic functions of four periods leads to the quaternary abelian group modulo p, where p is supposed to be an odd prime number. The equation for this p-section has two essentially distinct resolvents of degree $(p^4-1)/(p-1)$, as shown by JORDAN† and as follows incidentally in the present paper, §§ 2, 4 (Corollary), 16.

For the case $p=3$, the group arises in the problem of the 27 lines on a general cubic surface, as well as in the reduction of a binary sextic to CAYLEY's canonican form T^2-U^3.

The question of the existence of resolvents of degree lower than that mentioned above and the related, but more general, problem of the determination of all the subgroups of the abelian group form the subject of investigations now in progress by the writer. On account of the great complexity‡ of these problems only small values of p are being considered. The discussions for the various values of p have at least one question in common, that of the subgroups of order a power of p. To avoid duplication, this question is here treated for general p, together with a number of related questions.

2. The group of quaternary special abelian substitutions in the $GF[p^n]$ has a self-conjugate subgroup composed of the identity and the substitution which merely changes the sign of each variable. The quotient group G is simple except in the case $p=2$, $n=1$, when it is holoedrically isomorphic with the symmetric group on 6 letters.§ The order of G is

$$\tfrac{1}{2}p^{4n}(p^{4n}-1)(p^{2n}-1) \quad \text{if } p>2; \qquad p^{4n}(p^{4n}-1)(p^{2n}-1) \quad \text{if } p=2.$$

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† *Traité des Substitutions*, Note E, p. 666; Comptes Rendus (1870), pp. 326–328, 1028.

‡ Compare JORDAN, Note E, *loc. cit.*

§ *Linear Groups*, pp. 94–100.

The operators of G are conveniently designated as follows:

$$(1) \qquad \pm \begin{Vmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} \end{Vmatrix}.$$

When needed, the variables are designated $\xi_1, \eta_1, \xi_2, \eta_2$ in order.

An abelian substitution permutes the $p^{4n}-1$ letters $L_{\xi_1, \eta_1, \xi_2, \eta_2}$ in which $\xi_1, \eta_1, \xi_2, \eta_2$ are not simultaneously zero. Combining them into systems

$$\{L_{\mu\xi_1, \mu\eta_1, \mu\xi_2, \mu\eta_2}, \quad \text{where } \mu \text{ ranges over the marks } \neq 0 \text{ of the } GF[p^n]\},$$

we obtain $(p^{4n}-1)/(p^n-1)$ systems which are permuted amongst themselves by the *homogeneous* abelian substitutions. *Hence the group G may be represented as a (transitive †) substitution group on $(p^{4n}-1)/(p^n-1)$ letters.*

3. An operator of G is commutative with $L_{1,1}$ if and only if it be of the form

$$(2) \qquad \pm \begin{Vmatrix} 1 & k & a & c \\ 0 & 1 & 0 & 0 \\ 0 & \alpha c - \gamma a & \alpha & \gamma \\ 0 & \beta c - \delta a & \beta & \delta \end{Vmatrix} \qquad (\alpha\delta - \beta\gamma = 1).$$

The number of these operators is $p^{3n} \cdot (p^{2n}-1)p^n$. They form a subgroup $G_{p^{4n}(p^{2n}-1)}$. In view of their importance and frequent occurrence a notation for certain of these operators is introduced:

$$(3) \qquad [k, a, c, \gamma] = \pm \begin{Vmatrix} 1 & k & a & c \\ 0 & 1 & 0 & 0 \\ 0 & c - \gamma a & 1 & \gamma \\ 0 & -a & 0 & 1 \end{Vmatrix}.$$

They form a group $G_{p^{4n}}$ in view of the formula of composition

$$(4) \qquad \begin{aligned} &[K, A, C, \Gamma][k, a, c, \gamma] \\ &\quad = [K + k + aC - cA - aA\Gamma,\ A + a,\ C + c + a\Gamma,\ \Gamma + \gamma]. \end{aligned}$$

It follows that the second, third, fourth and fifth powers of (3) are respectively

$$[2k - a^2\gamma, 2a, 2c + a\gamma, 2\gamma], \ [3k - 4a^2\gamma, 3a, 3c + 3a\gamma, 3\gamma],$$

† American Journal of Mathematics, vol. 23 (1901), p. 367.

$$[4k - 10a^2\gamma,\ 4a,\ 4c + 6a\gamma,\ 4\gamma], \qquad [5k - 20a^2\gamma,\ 5a,\ 5c + 10a\gamma,\ 5\gamma].$$

By one-step induction, we readily find that

$$(5)\quad [k, a, c, \gamma]^r = [rk - \tfrac{1}{6}r(r^2-1)a^2\gamma,\ ra,\ rc + \tfrac{1}{2}r(r-1)a\gamma,\ r\gamma].$$

It follows that, if $p>3$, the pth power of $[k, a, c, \gamma]$ is $[0, 0, 0, 0]$, namely, identity; for $p = 2$ or 3, the power p^2 is the identity, while the pth power is the identity if and only if $a\gamma = 0$.

THEOREM. *For $p > 3$ the group $G_{p^{4n}}$ contains, in addition to the identity, only operators of period p. For $p = 2$ or 3, it contains only operators of periods 1, p, p^2.*

Since p^{4n} is the highest power of p which divides the order of G, there is a single set of conjugate subgroups of order p^{4n}.

COROLLARY.* *For $p > 3$, the group G contains no operator of period p^a, $a > 1$. For $p = 2$ or 3, G contains no operator of period p^a, $a > 2$.*

4. THEOREM. *Within G, the group $G_{p^{4n}}$ is self-conjugate only under the group*

$$(6)\quad G_{\frac{1}{2}p^{4n}(p^{2n}-1)^2} = (G_{p^{4n}},\ T_{1,a_1}T_{2,a_2}) : \pm \begin{Vmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11}^{-1} & 0 & 0 \\ 0 & \alpha_{11}^{-1}(\alpha_{22}\gamma_{12} - \alpha_{12}\gamma_{22}) & \alpha_{22} & \gamma_{22} \\ 0 & -\alpha_{11}^{-1}\alpha_{22}^{-1}\alpha_{12} & 0 & \alpha_{22}^{-1} \end{Vmatrix}.$$

The operator (1) transforms $[k, a, c, \gamma]$ into an operator T replacing η_1 by

$$A_2\xi_1 + B_2\eta_1 + C_2\xi_2 + D_2\eta_2,$$

where, after reductions by means of the abelian conditions on (1),

$$A_2 = -k\beta_{11}^2 - \gamma\beta_{12}^2 + 2a\beta_{11}\delta_{12} - 2c\beta_{11}\beta_{12} + a\gamma\beta_{11}\beta_{12},$$

$$B_2 = 1 + k\alpha_{11}\beta_{11} + \gamma\alpha_{12}\beta_{12} - a\beta_{11}\gamma_{12} - a\alpha_{11}\delta_{12} + c\alpha_{11}\beta_{12} + c\beta_{11}\alpha_{12} - a\gamma\alpha_{11}\beta_{12},$$

$$C_2 = -k\beta_{11}\beta_{21} - \gamma\beta_{12}\beta_{22} + a\beta_{11}\delta_{22} + a\delta_{12}\beta_{21} - c\beta_{11}\beta_{22} - c\beta_{12}\beta_{21} + a\gamma\beta_{12}\beta_{21},$$

$$D_2 = k\beta_{11}\alpha_{21} + \gamma\beta_{12}\alpha_{22} - a\beta_{11}\gamma_{22} - a\delta_{12}\alpha_{21} + c\beta_{11}\alpha_{22} + c\alpha_{21}\beta_{12} - a\gamma\alpha_{21}\beta_{12}.$$

Since T shall belong to $G_{p^{4n}(p^{2n}-1)}$, we must have

$$A_2 = 0, \quad B_2 = 1, \quad C_2 = 0, \quad D_2 = 0,$$

for every set of marks k, a, c, γ of the $GF[p^n]$. But a linear homogeneous function of $k, a, c, \gamma, a\gamma$ with coefficients in a field F of order >1 equals zero for

*Compare Transactions, vol. 2 (1901), p. 113. The signs of Z and W in the footnote should be changed.

every mark k, a, c, γ of F if and only if each of its coefficients is zero. Hence $A_2 = 0$ requires that $\beta_{11} = \beta_{12} = 0$; then $B_2 = 1$ requires that $\alpha_{11}\delta_{12} = 0$; $C_2 = 0$ requires that $\delta_{12}\beta_{21} = 0$; $D_2 = 0$ requires that $\delta_{12}\alpha_{21} = 0$. If $\delta_{12} \neq 0$, then all the coefficients in the first column of (1) are zero. Hence $\delta_{12} = 0$.

Applying the abelian conditions, we find that (1) becomes

$$(7) \qquad \pm \begin{Vmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11}^{-1} & 0 & 0 \\ 0 & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & \beta_{22} & \delta_{22} \end{Vmatrix}, \quad \begin{pmatrix} \gamma_{21} = \alpha_{11}^{-1}(\alpha_{22}\gamma_{12} - \alpha_{12}\gamma_{22}) \\ \delta_{21} = \alpha_{11}^{-1}(\beta_{22}\gamma_{12} - \alpha_{12}\delta_{22}) \\ \alpha_{22}\delta_{22} - \beta_{21}\gamma_{22} = 1 \end{pmatrix}$$

The latter is seen to transform $[k, a, c, \gamma]$ into

$$\pm \begin{Vmatrix} 1 & k' & a\alpha_{11}\delta_{22} - c\alpha_{11}\beta_{22} - \gamma\alpha_{12}\beta_{22} & c' \\ 0 & 1 & 0 & 0 \\ 0 & c\alpha_{11}\alpha_{22} - a\alpha_{11}\gamma_{22} - a\gamma\alpha_{11}\alpha_{22} + \gamma\alpha_{12}\alpha_{22} & 1 - \gamma\alpha_{22}\beta_{22} & \gamma\alpha_{22}^2 \\ 0 & c\alpha_{11}\beta_{22} - a\alpha_{11}\delta_{22} - a\gamma\alpha_{11}\beta_{22} + \gamma\alpha_{12}\beta_{22} & -\gamma\beta_{22}^2 & 1 + \gamma\alpha_{22}\beta_{22} \end{Vmatrix},$$

where

$$k' = k\alpha_{11}^2 - 2a\alpha_{11}\gamma_{12} + 2c\alpha_{11}\alpha_{12} - a\gamma\alpha_{11}\alpha_{12} + \gamma\alpha_{12}^2, \quad c' = c\alpha_{11}\alpha_{22} - a\alpha_{11}\gamma_{22} + \gamma\alpha_{12}\alpha_{22}.$$

For $\gamma \neq 0$, this operator is of the form (2) with $\beta = 0$ only when $\beta_{22} = 0$. Inversely, if $\beta_{22} = 0$, it has the form $[k', a\alpha_{11}\delta_{22}, c', \gamma\alpha_{22}^2]$.

Since (7) is the product of an operator (2) by $T_{1, a_{11}}$ and since the latter transforms every operator (2) into an operator (2), we derive

COROLLARY I. *Within G, the group $G_{p^{4n}(p^{2n}-1)}$ is self-conjugate only under the group $G_{\frac{1}{2}p^{4n}(p^{2n}-1)(p^n-1)} = (G_{p^{4n}(p^{2n}-1)}, T_{1,a})$ of the operators (7). The latter is therefore one of $(p^{4n}-1)/(p^n-1)$ conjugate subgroups of G.*

COROLLARY II. *Within G, the group $G_{\frac{1}{2}p^{4n}(p^n-1)^2}$ is self-conjugate only under itself and hence is one of $(p^{2n}+1)(p^n+1)^2$ conjugate subgroups.*

Distribution of the operators of $G_{p^{4n}}$ into sets of conjugates.

5. Upon specialization of (7) by setting $\alpha_{11} = 1$, $\alpha_{22} = \delta_{22} = 1$, $\beta_{22} = 0$, it follows from the preceding section that $[p, q, r, s]$ transforms $[k, a, c, \gamma]$ into

$$[k + 2qc - 2ra - qa\gamma + q^2\gamma, a, c + q\gamma - sa, \gamma].$$

In order that the latter shall be identical with $[k, a, c, \gamma]$ for every p, q, r, s, it is necessary and sufficient that $a = \gamma = c = 0$ if $p > 2$, and that $a = \gamma = 0$

if $p=2$. *Hence the only self-conjugate operators in* $G_{p^{4n}}$ *are* $[k, 0, 0, 0]$ *if* $p>2$, *but are* $[k, 0, c, 0]$ *if* $p=2$.

Henceforth let $p>2$. Let a and γ have fixed values each $\neq 0$. Then $[k, a, c, \gamma]$ is conjugate with $[k', a, c', \gamma]$ if and only if

$$k' = k + 2qc - 2ra - qa\gamma + q^2\gamma, \qquad c' = c + q\gamma - sa$$

have solutions p, q, r, s in the field. The second equation can be satisfied by a suitable choice of q or s, and the first by a subsequent choice of r. *Hence the operators* $[k, a, c, \gamma]$, $a \neq 0, \gamma \neq 0$, *fall into* $(p^n-1)^2$ *sets* $S_{a,\gamma}$ *each of* p^{2n} *conjugate operators within* $G_{p^{4n}}$.

An operator $[k, a, c, 0]$, $a \neq 0$, is conjugate with $[k', a, c', 0]$ if and only if there exist solutions q, r, s of

$$k' = k + 2qc - 2ra, \qquad c' = c - sa.$$

We may take $q=0$ and determine r and s by the first and second equations, respectively. *Hence the operators* $[k, a, c, 0]$, $a \neq 0$, *fall into* p^n-1 *sets* S_a *each of* p^{2n} *conjugate operators.*

Similarly, *the operators* $[k, 0, c, 0]$, $c \neq 0$, *fall into* p^n-1 *sets* S'_c *each of* p^n *conjugate operators.*

There remain the operators $[k, 0, c, \gamma]$, $\gamma \neq 0$. The latter is conjugate only with $[k', 0, c', \gamma]$, where $k' = k + 2qc + q^2\gamma$, $c' = c + q\gamma$. Hence

$$q = \frac{1}{\gamma}(c' - c), \qquad k' = k + \frac{1}{\gamma}(c'^2 - c^2).$$

Hence if k, c, c', γ are any given marks such that $\gamma \neq 0$, there exists an unique mark k' for which $[k, 0, c, \gamma]$ and $[k', 0, c', \gamma]$ are conjugate. *Hence the operators* $[k, 0, c, \gamma]$, $\gamma \neq 0$, *fall into* $p^n(p^n-1)$ *sets* $\Sigma_{k,\gamma}$ *each of* p^n *conjugate operators.*

For a set $S_{a,\gamma}$, S_a or S'_c, the subscripts are defining invariants. But for $\Sigma_{k,\gamma}$, the subscript γ alone is an invariant.

THEOREM. *The operators of* $G_{p^{4n}}$ *fall into* $2p^{2n}-1$ *distinct sets of conjugate operators;* p^n *sets contain each a single operator,* $p^{2n}-p^n$ *sets contain each* p^{2n} *operators,* $p^{2n}-1$ *sets contain each* p^n *operators.*

6. In illustration of the results of § 5, consider the important case $p^n=3$. Then, by § 3, $[k, a, c, \gamma]^3 = [0, 0, 0, 0]$ = identity if and only if $a\gamma = 0$. Hence the 44 operators of period 3 in G_{81} are $[k, a, c, 0]$ and $[k, 0, c, \gamma]$, excluding $[0, 0, 0, 0]$. The remaining 36 operators $[k, a, c, \gamma]$, $a\gamma \neq 0$, are of period 9. The 81 operators fall into the following 17 sets of conjugates: *

* The last six sets may be given compactly as follows

$$\{[k, 0, 1, \gamma], [k, 0, -1, \gamma], [k-\gamma, 0, 0, \gamma]\} \quad (k=0, 1, -1; \gamma=1, -1).$$

$$[0,0,0,0];\quad [1,0,0,0];\quad [-1,0,0,0];$$

$$[k,1,c,1];\quad [k,1,c,-1];\quad [k,-1,c,1];\quad [k,-1,c,-1];$$

$$[k,1,c,0];\quad [k,-1,c,0];\quad [k,0,1,0];\quad [k,0,-1,0];$$

$$\{[0,0,1,\pm 1],[0,0,-1,\pm 1],[\mp 1,0,0,\pm 1]\};$$

$$\{[0,0,0,\pm 1],[\pm 1,0,1,\pm 1],[\pm 1,0,-1,\pm 1]\};$$

$$\{[\pm 1,0,0,\pm 1],[\mp 1,0,1,\pm 1],[\mp 1,0,-1,\pm 1]\},$$

where the upper (or lower) signs belong together, while the operators given by $k, c = 0, 1, -1$ belong to the same set.

Determination of the self-conjugate subgroups of $G_{p^{4n}}$.

7. If a self-conjugate subgroup H contains one operator of a set $S_{a,\gamma}$, $a \neq 0, \gamma \neq 0$, it contains every $[k, a, c, \gamma]$, k and c being arbitrary. By (5), H contains one, and hence every, operator $[k, ra, c, r\gamma]$, for each integer $r = 1, 2, \cdots, p-1$. But, by (4),

$$[K, Ra, C, R\gamma][k, ra, c, r\gamma]$$

$$= [K + k + raC - Rac - R^2ra^2\gamma, (R+r)a, C + c + Rra\gamma, (R+r)\gamma].$$

It follows first that H contains also every $[k, 0, c, 0]$ and second that H contains the *group*

$$(8)\quad H^{a,\gamma}_{p^{2n+1}} = \{[k, ra, c, r\gamma], (r = 0, 1, 2, \cdots, p-1;\ k \text{ and } c \text{ arbitrary})\}.$$

Note that (a, γ) and (a', γ') define the same group if $a'/a = \gamma'/\gamma =$ an integer prime to p, so that there are $p^{n-1}(p^n - 1)$ distinct groups (8).

If H contains one operator of a set S_a, $a \neq 0$, it contains $[k, a, c, 0]$ for k and c arbitrary. Then, by (5), H contains every $[k, ra, c, 0]$, $r = 1, \cdots, p-1$. By (4),

$$[K, Ra, C, 0][k, ra, c, 0] = [K + k + raC - Rac, (R+r)a, C + c, 0].$$

Hence H contains also every $[k, 0, c, 0]$, and contains the *group*

$$(9)\quad G^{a}_{p^{2n+1}} = \{[k, ra, c, 0], (r = 0, 1, \cdots, p-1;\ k \text{ and } c \text{ arbitrary})\}.$$

If H contains one operator of a set S'_c, $c \neq 0$, it contains $[k, 0, c, 0]$ for k arbitrary. The rth power of the latter is $[rk, 0, rc, 0]$. Also

$$[K, 0, Rc, 0][k, 0, rc, 0] = [K + k, 0, Rc + rc, 0].$$

Hence H contains the *commutative group*

$$(10) \qquad K^{r}_{p^{n+1}} = \{[k, 0, rc, 0], (r = 0, 1, \cdots, p-1;\ k \text{ arbitrary})\}.$$

If H contains $[k, 0, 0, 0]$, it contains the cyclic group

$$(11) \qquad C^{k}_{p} = \{[rk, 0, 0, 0], (r = 0, 1, \cdots, p-1)\}.$$

Finally, let H contain one $[k, 0, c, \gamma]$, $\gamma \neq 0$. By (4),

$$(12) \qquad [K, 0, C, \Gamma][k, 0, c, \gamma] = [K + k, 0, C + c, \Gamma + \gamma].$$

Now $[k, 0, c, \gamma]$ is conjugate with $[k + (c'^2 - c^2)/\gamma, 0, c', \gamma]$, c' being arbitrary. Multiplying the latter by the inverse of the former and setting $c' - c = \lambda$, we get $A = [(\lambda^2 + 2\lambda c)/\gamma, 0, \lambda, 0]$, which therefore occurs in H for λ arbitrary. Replacing λ by $-\lambda$, and taking the product of the resulting operator by A, we get $[2\lambda^2/\gamma, 0, 0, 0]$. Hence, if $p > 2$, H contains every $[k, 0, 0, 0]$, since $\lambda_1^2 - \lambda_2^2$ can be made to take any desired value, so that H contains every $[k, 0, c, 0]$. The same result follows for any p by § 5, since A is conjugate with every $[k, 0, \lambda, 0]$. By (5),

$$[k, 0, c, \gamma]^r = [rk, 0, rc, r\gamma].$$

Hence H contains the *commutative group*

$$(13) \qquad K^{\gamma}_{p^{n+1}} = \{[k, 0, c, r\gamma], (r = 0, 1, \cdots, p-1;\ k \text{ and } c \text{ arbitrary})\}:$$

THEOREM. *The $p^{n-1}(p^n - 1)$ groups* (8), *the p^{n-1} groups* (9), *the p^{n-1} groups* (10), *the p^{n-1} groups* (11) *in addition to the identity, and the p^{n-1} groups* (13), *together with all groups resulting from their combination, give all the self-conjugate subgroups of $G_{p^{4n}}$.*

8. For $n = 1$, the complete list of self-conjugate subgroups follows without further computation. We may now drop one of the superscripts in the notation for each group, giving the following groups:*

$$(8') \qquad H'_{p^3} = \{[k, ta, c, a],\ (k, a, c = 0, 1, \cdots, p-1)\} \qquad (t = 1, 2, \cdots, p-1),$$

$$(9') \qquad G_{p^3} = \{[k, a, c, 0],\ (k, a, c = 0, 1, \cdots, p-1)\},$$

$$(10') \qquad K_{p^2} = \{[k, 0, c, 0],\ (k, c = 0, 1, \cdots, p-1)\},$$

$$(11') \qquad C_{p} = \{[k, 0, 0, 0],\ (k = 0, 1, \cdots, p-1)\},$$

$$(13') \qquad K_{p^3} = \{[k, 0, c, \gamma],\ (k, c, \gamma = 0, 1, \cdots, p-1)\}.$$

* In (8), only the ratio a/γ is now invariant. It is set equal to t.

Evidently C_p lies in all these groups, while K_{p^2} lies in all of order p^3. Hence a combination of two or more of these groups leads to no new group other than G_{p^3}. We may therefore state the

THEOREM. *The group G_{p^3} contains only the self-conjugate subgroups* (8′)–(13′) *in addition to itself and the identity.*

Since a subgroup $G_{p^{n-1}}$ of G_{p^n} is necessarily self-conjugate, we have the

COROLLARY. *The $p+1$ groups* (8′), (9′), (13′) *give all the subgroups of order p^3 of G_{p^3}.*

9. THEOREM. *Within G for $n=1$, every subgroup of order p^3 is conjugate with* * *H_{p^3}, G_{p^3} or K_{p^3}, while no two of the latter are conjugate.*

Within G, every subgroup of order p^3 is contained† in one or more subgroups of order p^4, the latter being conjugate with G_{p^4} by SYLOW's theorem. We may therefore confine our attention to the groups (8′), (9′) and (13′). But $T_{1,t^{-1}}$ transforms $[k, ta, c, a]$ into $[kt^{-2}, a, t^{-1}c, a]$ and hence transforms H'_{p^3} into $H'^{1}_{p^3} \equiv H_{p^3}$. In view of the formulæ

$$[K, A, C, A]\,[k, a, c, a] = [K+k+aC-cA-aA^2,\ A+a,\ C+c+aA,\ A+a], \tag{14}$$

$$[K, A, C, 0]\,[k, a, c, 0] = [K+k+aC-cA,\ A+a,\ C+c,\ 0], \tag{15}$$

the only self-conjugate operators in either H_{p^3} or G_{p^3} are $[k, 0, 0, 0]$. Hence neither is conjugate with the commutative group K_{p^3}. The first two are not conjugate by §§ 14–15. For $p=3$, this also follows from the fact that H_{27} contains operators of period 9, while G_{27} and K_{27} contain only operators of periods 1 and 3.

Conjugacy of the operators of H_{p^3}; its self-conjugate subgroups.

10. By § 5, $[p, q, r, q]$ transforms $[k, a, c, a]$ into $[k', a, c, a]$, where

$$k' = k + 2qc - 2ra - qa^2 + q^2a.$$

Let k, a, c be given. If $a \neq 0$, we may take $q=0$ and choose r to make k' take any assigned value. If $a=0$, $c \neq 0$, we can choose q to make k' arbitrary. *Hence the operators of H_{p^3} fall into exactly p^2+p-1 sets of conjugates:*

$$\{[k, a, c, a],\ k=0, 1, \cdots, p-1\} \quad (a, c \text{ fixed integers not both} \equiv 0),$$

$$\{[k, 0, 0, 0]\} \quad (k \text{ a fixed integer}).$$

* For $t=1$, (8′) is designated also H_{p^3}.

† BURNSIDE, *The Theory of Groups*, p. 94, Corollary III

11. If a self-conjugate subgroup J contains the operator $[k', a, c, a]$, where a and c are not both $\equiv 0$, it contains every $[k, a, c, a]$, $k = 0, 1, \cdots, p-1$. By (14), the inverse of the latter is $[-k, -a, -c+a^2, -a]$. The product of the latter by $[K, a, c, a]$ is $[K-k, 0, 0, 0]$. Hence J contains the commutative group generated by $[1, 0, 0, 0]$ and $[0, a, c, a]$:

$$(16)\quad H_{p^2}^{a,c} = \{[k, ra, rc + \tfrac{1}{2}r(r-1)a^2, ra], (k, r = 0, 1, \cdots, p-1)\}.$$

For $a = 0$, the group is (10′). For $p = 3$, $a \neq 0$, the group is a cyclic group of order 9 since $[0, a, c, a]^3 = [-a^3, 0, 0, 0]$.

THEOREM. *For $p > 2$ the group H_{p^3} has only the following self-conjugate subgroups in addition to itself, the identity and C_p:*

if $p > 3$, K_{p^2} and p further non-cyclic groups (16) with $a \neq 0$;

if $p = 3$, K_9 and 3 cyclic groups of order 9 generated by respectively $[0, 1, 1, 1]$, $[1, -1, 1, -1]$ and $[1, 1, -1, 1]$.

COROLLARY. *For $p > 2$, H_{p^3} has exactly $p + 1$ subgroups of order p^2.*

Conjugacy of the operators of G_{p^3}; its self-conjugate subgroups.

12. By § 5, $[p, q, r, 0]$ transforms $[k, a, c, 0]$ into $[k', a, c, 0]$, where $k' = k + 2qc - 2ra$. If a and c are fixed marks on the $GF[p]$, $p > 2$, not both zero, we can choose q and r to make k' take any assigned value. But if $a = c = 0$, then $[k, 0, 0, 0]$ is self-conjugate.

Hence the operators of G_{p^3} fall into exactly $p^2 + p - 1$ sets of conjugates:

$$\{[k, a, c, 0], k = 0, 1, \cdots, p-1\} \quad (a, c, \textit{fixed integers not both} \equiv 0),$$

$$\{[k, 0, 0, 0]\} \quad (k \textit{ a fixed integer}).$$

13. If a self-conjugate subgroup J of G_{p^3} contains one operator $[k, a, c, 0]$, where a and c are fixed integers not both zero, it contains them all. Also J contains the inverse $[-k, -a, -c, 0]$ [see formula (14)]. The product of the latter by $[K, a, c, 0]$ is $[K-k, 0, 0, 0]$. Hence J contains the commutative group generated by $[1, 0, 0, 0]$ and $[0, a, c, 0]$:

$$(17)\qquad K_{p^2}^{a,c} = \{[k, ra, rc, 0], (k, r = 0, 1, \cdots, p-1)\}.$$

For $a = 0$, the group is (10′). Whether $p = 3$ or $p > 3$, the groups are non-cyclic.

THEOREM. *For $p \neq 2$, the group G_{p^3} has as its self-conjugate subgroups, besides itself and the identity, the group C_p and $p + 1$ non-cyclic commutative groups (17). It thus has exactly $p + 1$ subgroups of order p^2.*

Largest subgroups in which G_{p^3}, H_{p^3}, K_{p^3} are self-conjugate.

14. THEOREM. *Within $G_{n=1}$, the group G_{p^3} is self-conjugate only under the group $G_{\frac{1}{2}p^4(p^2-1)(p-1)}$ of the operators (7) with coefficients modulo p.*

Proceeding as in § 4 with $\gamma = 0$, we must have $A_2 = 0$, $B_2 = 1$, $C_2 = 0$, $D_2 = 0$ for every set of integers k, a, c. Hence

$$\beta_{11} = 0,\ \alpha_{11}\delta_{12} = 0,\ \alpha_{11}\beta_{12} = 0;\ \delta_{12}\beta_{21} = 0,\ \beta_{12}\beta_{21} = 0;\ \delta_{12}\alpha_{21} = 0,\ \beta_{12}\alpha_{21} = 0.$$

If $\beta_{12} \neq 0$, then $\alpha_{11} = \beta_{21} = \alpha_{21} = 0$, so that all the coefficients in the first column of (1) are zero; hence must $\beta_{12} = 0$. Similarly, $\delta_{12} = 0$. Hence (1) reduces to (7). The latter transforms $[k, a, c, 0]$ into $[k', a', c', 0]$, where, by § 4,

$$k' = k\alpha_{11}^2 - 2a\alpha_{11}\gamma_{12} + 2c\alpha_{11}\alpha_{12},\quad a' = a\alpha_{11}\delta_{22} - c\alpha_{11}\beta_{22},\quad c' = c\alpha_{11}\alpha_{22} - a\alpha_{11}\gamma_{22}.$$

For general n, the conditions that k', a', c' shall be integers modulo p, $p > 2$, for every set of integers k, a, c, are that α_{11}^2, $\alpha_{11}\gamma_{12}$, $\alpha_{11}\alpha_{12}$, $\alpha_{11}\delta_{22}$, $\alpha_{11}\beta_{22}$, $\alpha_{11}\alpha_{22}$ and $\alpha_{11}\gamma_{22}$ shall be integral marks. The same is then true for $\alpha_{11}\gamma_{21}$ and $\alpha_{11}\delta_{21}$. Hence the ratios of the various coefficients of (7) to α_{11} must belong to the $GF[p]$. Hence (7) is the product of an operator with integral coefficients by $T_{1,\alpha_{11}}T_{2,\alpha_{11}}$, where α_{11}^2 is integral. We may restrict α_{11} to the roots of $x^2 = \nu$, where ν is a particular non-residue of p. We have therefore the

COROLLARY. *For $p > 2$ and n even, the group G_{p^3} is self-conjugate within G only under the group $G_{p^{4(p^2-1)(p-1)}}$ of operators (7) whose coefficients are all integers or all integral multiples of a square root of a non-residue of p. For n odd, the group is that in the theorem.*

15. THEOREM. *Within $G_{n=1}$, the group H_{p^3} is self-conjugate only under*

$$G_{\frac{1}{2}p^4(p-1)} = (G_{p^3},\ T_{1,\alpha^2}T_{2,\alpha}). \tag{18}$$

Proceeding as in § 4 with $\gamma = a$, we must have $A_2 = 0$, $B_2 = 1$, $C_2 = 0$, $D_2 = 0$ for every set of integers k, c, a. If a linear homogeneous function of k, c, a, a^2 is zero for every k, c, a, then its coefficients are all zero. The conditions are therefore $\beta_{11} = \beta_{12} = 0$, $\alpha_{11}\delta_{12} = 0$, $\beta_{21}\delta_{12} = 0$, $\alpha_{21}\delta_{12} = 0$, respectively. Hence $\delta_{12} = 0$, so that the transformer must be of the form (7). Then, as in § 4, $\beta_{22} = 0$, and $[k, a, c, a]$ is transformed into $[k', a\alpha_{11}\delta_{22}, c', a\alpha_{22}^2]$. Hence must $\alpha_{11}\delta_{22} = \alpha_{22}^2$. But $\alpha_{22}\delta_{22} = 1$. Hence $\alpha_{11} = \alpha_{22}^3$.

16. THEOREM. *Within $G_{n=1}$, the group K_{p^3} is self-conjugate only under the group $H_{\frac{1}{2}p^4(p^2-1)(p-1)}$ of the operators*

$$\pm \begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \delta_{11} & 0 & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & 0 & \delta_{22} \end{bmatrix}, \quad \begin{pmatrix} \alpha_{11}\delta_{11} + \alpha_{12}\delta_{12} = 1,\ \alpha_{11}\delta_{21} + \alpha_{12}\delta_{22} = 0 \\ \alpha_{22}\delta_{22} + \alpha_{21}\delta_{21} = 1,\ \alpha_{22}\delta_{12} + \alpha_{21}\delta_{11} = 0 \\ \alpha_{11}\gamma_{21} + \alpha_{12}\gamma_{22} - \alpha_{21}\gamma_{11} - \alpha_{22}\gamma_{12} = 0 \end{pmatrix}, \tag{19}$$

and hence is one of $(p^4 - 1)/(p - 1)$ conjugate subgroups.

Proceeding as in § 4 with $a = 0$, we find that $A_2 = 0$ for every k, c, γ requires that $\beta_{11} = \beta_{12} = 0$. In view of these values, we have $B_2 = 1$, $C_2 = 0$, $D_2 = 0$, identically. By computation, the operator (1) with $\beta_{11} = \beta_{12} = 0$ is seen to transform $[k, 0, c, \gamma]$ into

$$(20) \qquad \pm \begin{Vmatrix} 1 & K & A & C \\ 0 & 1 & 0 & 0 \\ 0 & C & 1-E & \Gamma \\ 0 & -A & B & 1+E \end{Vmatrix},$$

$$(21) \quad \begin{cases} K = k\alpha_{11}^2 + \gamma\alpha_{12}^2 + 2c\alpha_{11}\alpha_{12}, & A = -k\alpha_{11}\beta_{21} - c\alpha_{11}\beta_{22} - c\alpha_{12}\beta_{21} - \gamma\alpha_{12}\beta_{22}, \\ \Gamma = k\alpha_{21}^2 + \gamma\alpha_{22}^2 + 2c\alpha_{21}\alpha_{22}, & C = k\alpha_{11}\alpha_{21} + c\alpha_{11}\alpha_{22} + c\alpha_{12}\alpha_{21} + \gamma\alpha_{12}\alpha_{22}, \\ B = -k\beta_{21}^2 - \gamma\beta_{22}^2 - 2c\beta_{21}\beta_{22}, & E = k\alpha_{21}\beta_{21} + c\alpha_{22}\beta_{21} + c\alpha_{21}\beta_{22} + \gamma\alpha_{22}\beta_{22}. \end{cases}$$

In order that (20) shall reduce to $[K, 0, C, \Gamma]$, it is necessary that $\beta_{21} = \beta_{22} = 0$ (from $B = 0$) and sufficient (since then $A = 0$, $E = 0$). Then the transformer becomes (19) on applying the abelian conditions. The first four of the resulting conditions (19) may be solved and given the equivalent form

$$(22) \quad \alpha_{11} = \delta_{22}/\Delta, \quad \alpha_{12} = -\delta_{21}/\Delta, \quad \alpha_{21} = -\delta_{12}/\Delta, \quad \alpha_{22} = \delta_{11}/\Delta, \quad \Delta \equiv \delta_{11}\delta_{22} - \delta_{12}\delta_{21}.$$

Evidently Δ is $\not\equiv 0$ since it is a factor of the determinant of (19). The number of sets of integers δ_{ij} modulo p for which $\Delta \not\equiv$ is $(p^2-1)(p^2-p)$. The fifth condition (19) involves the γ_{ij} linearly with coefficients not all zero, so that three of the γ_{ij} are arbitrary. The number of homogeneous substitutions (19) is therefore $p^3(p^2-1)(p^2-p)$.

Subgroups of order p^2 of the commutative group K_{p^3}.

17. Such a subgroup contains, in addition to the identity, only operators of period p. Hence it is generated by two commutative operators A and B of period p. The quotient of $(p^3-1)(p^3-p)$, the number of ways A and B may be selected, by $(p^2-1)(p^2-p)$, the number of ways a given group (A, B) can be generated, gives *the number p^2+p+1 of distinct subgroups of order p^2*.

Now $[p, q, r, s]$ transforms the general operator $[k, 0, c, \gamma]$ of K_{p^3} into $[k', 0, c', \gamma]$, where

$$k' = k + 2qc + q^2\gamma, \qquad c' = c + q\gamma.$$

If $\gamma \not\equiv 0$, we can choose q to make $c' = 0$. By (5), the rth power of $[k', 0, 0, \gamma]$ is $[rk', 0, 0, r\gamma]$. Hence we can choose $[k, 0, 0, 1]$ as the first generator. If $\gamma = 0$, $c \not\equiv 0$, and if $p > 2$, we can choose q to make

$k' = 0$. A suitable power of $[0, 0, c, 0]$ gives $[0, 0, 1, 0]$. Finally, if $\gamma = c = 0$, we are led to $[1, 0, 0, 0]$.* With $A = [k, 0, 0, 1]$, the group contains a second generator $[k', 0, c, 0]$. We may therefore suppose that the first generator is $[1, 0, 0, 0]$ or $[0, 0, 1, 0]$.

For $A = [1, 0, 0, 0]$, we may take $[0, 0, c, \gamma]$ as the second generator. If $\gamma = 0$, a suitable power gives $B = [0, 0, 1, 0]$. If $\gamma \neq 0$, a suitable power gives $[0, 0, c, 1]$. But $[p, q, r, s]$ transforms A into itself and $[0, 0, c, 1]$ into $[2qc + q^2, 0, c + q, 1]$. Taking $q = -c$, we are led to $B = [0, 0, 0, 1]$. The resulting groups are

$$(23) \qquad K_{p^2} = \{[k, 0, c, 0]\}, \qquad K^*_{p^2} = \{[k, 0, 0, \gamma]\}.$$

For $A = [0, 0, 1, 0]$, we may take $[k, 0, 0, \gamma]$ as the second generator. The case $\gamma = 0$ leads to K_{p^2}. For $\gamma \neq 0$, a suitable power gives $[\sigma, 0, 0, 1]$. If $\sigma = 0$, the resulting group $\{[0, 0, c, \gamma]\}$ is transformed by P_{12} into $\{[\gamma, 0, c, 0]\} = K_{p^2}$. The case $\sigma \neq 0$ requires detailed study.

We proceed to determine whether or not a group $\{[\sigma\gamma, 0, c, \gamma]\}$ is conjugate within $G_{n=1}$ with another such group or with one of the groups (23). If (1) transforms $[\sigma\gamma, 0, c, \gamma]$ into an operator T leaving η_1 fixed, then (as in § 4 with $k = \sigma\gamma$, $a = 0$)

$$A_2 \equiv -\sigma\gamma\beta_{11}^2 - \gamma\beta_{12}^2 - 2c\beta_{11}\beta_{12} = 0$$

for every c and γ. Hence $\beta_{11}\beta_{12} = 0$, $\sigma\beta_{11}^2 + \beta_{12}^2 = 0$, so that $\beta_{11} = \beta_{12} = 0$. We may therefore proceed as in § 16, with $k = \sigma\gamma$, and impose the conditions $A = B = E$ that T shall reduce to the form $[K, 0, C, \Gamma]$. Now

$$B \equiv -\sigma\gamma\beta_{21}^2 - \gamma\beta_{22}^2 - 2c\beta_{21}\beta_{22} = 0$$

for every c and γ requires that $\beta_{21} = \beta_{22} = 0$. Hence (1) must become (19).

The condition $\Gamma = 0$ for every c and γ requires $\alpha_{21} = \alpha_{22} = 0$, which is impossible. *Hence a group* $\{[\sigma\gamma, 0, c, \gamma]\}$ *is not conjugate with* K_{p^2}.

* For the problem of the conjugacy of the cyclic subgroups of order p of K_{p^2}, we note that we may restrict the generators to $[1, 0, 0, 0]$, $[0, 0, 1, 0]$, $[\tau, 0, 0, 1]$, where $-\tau$ is a particular quadratic non-residue of p. Indeed, (19) transforms $[k, 0, 0, 1]$ into $K, 0, C, \Gamma]$, where

$$K = ka_{11}^2 + a_{12}^2, \qquad C = ka_{11}a_{21} + a_{12}a_{22}, \qquad \Gamma = ka_{21}^2 + a_{22}^2.$$

But for any set of integers a_{ij} such that $\Delta_1 \equiv a_{11}a_{22} - a_{12}a_{21}$ is $\neq 0$, there exists a substitution (19). If $-k$ is a quadratic residue or 0, the conditions $K = \tau\Gamma$, $C = 0$, $\Delta_1 \neq 0$ become

$$kx^2 + z^2 - \tau ky^2 - \tau = 0, \qquad kxy + z = 0, \qquad x - yz \neq 0,$$

upon setting $a_{11} = xa_{22}$, $a_{21} = ya_{22}$, $a_{12} = za_{22}$, and noting that $a_{22} \neq 0$. Eliminating z, we get

$$(kx^2 - \tau)(ky^2 + 1) = 0, \qquad x(1 + ky^2) \neq 0.$$

Hence the conditions can be satisfied if and only if τ/k is a residue, i. e., if $-\tau$ is a non-residue. A suitable power of $[\tau\Gamma, 0, 0, \Gamma]$ gives $[\tau, 0, 0, 1]$.

The condition $C = 0$ for every c and γ requires

$$\sigma\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0, \qquad \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} = 0. \tag{24}$$

Taking α_{11} and α_{12} as the unknowns, not both of which are zero by (19), we see that the determinant $\sigma\alpha_{21}^2 - \alpha_{22}^2$ is zero. Taking α_{21} and α_{22} as the unknowns, the determinant $\sigma\alpha_{11}^2 - \alpha_{12}^2 = 0$. *Hence σ must be a square.* With this condition satisfied, we have $\alpha_{22} = s\alpha_{21}$, $\alpha_{12} = -s\alpha_{11}$, the sign following from either condition (24). The abelian conditions (19) then give

$$\delta_{21} = s\delta_{22}, \qquad \delta_{11} = -s\delta_{12}, \qquad -2s\alpha_{11}\delta_{12} = 1, \qquad 2s\alpha_{21}\delta_{22} = 1,$$

with a linear relation on the γ_{ij}. *The resulting $p^3(p-1)^2$ operators of $G_{n=1}$ transform $\{[s^2\gamma, 0, c, \gamma]\}$ into $K_{p^2}^*$.*

In order that (19) shall transform

$$\{[\sigma\gamma, 0, c, \gamma]\} \text{ into } \{[\sigma_1\gamma_1, 0, c_1, \gamma_1]\}$$

where σ and σ_1 are not-squares, it is necessary and sufficient that $K = \sigma_1\Gamma$, viz.,

$$\sigma\gamma\alpha_{11}^2 + \gamma\alpha_{12}^2 + 2c\alpha_{11}\alpha_{12} = \sigma_1(\sigma\gamma\alpha_{21}^2 + \gamma\alpha_{22}^2 + 2c\alpha_{21}\alpha_{22}),$$

for every c and γ. The conditions are

$$\alpha_{11}\alpha_{12} = \sigma_1\alpha_{21}\alpha_{22}, \qquad \sigma\alpha_{11}^2 + \alpha_{12}^2 = \sigma_1\sigma\alpha_{21}^2 + \sigma_1\alpha_{22}^2.$$

Squaring each, we find that

$$\sigma\alpha_{11}^2 - \alpha_{12}^2 = \pm\sigma_1(\sigma\alpha_{21}^2 - \alpha_{22}^2).$$

Hence $2\sigma\alpha_{11}^2$ equals either $2\sigma_1\sigma\alpha_{21}^2$ or else $2\sigma_1\alpha_{22}^2$. The first case is excluded since σ_1 is a not-square and α_{11} and α_{21} are not both zero in view of the abelian conditions. Hence $\alpha_{22} = s\alpha_{11}$, where $s^2 = \sigma/\sigma_1$. Whether α_{11} is zero or not, we have $\alpha_{12} = s\sigma_1\alpha_{21}$. Since

$$\Delta_1 = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = s\alpha_{11}^2 - s\sigma_1\alpha_{21}^2 \neq 0,$$

the δ_{ij} are uniquely determined in terms of the α_{ji} by the conditions (19):

$$\delta_{22} = \alpha_{11}/\Delta_1, \quad \delta_{21} = -s\sigma_1\alpha_{21}/\Delta_1, \quad \delta_{12} = -\alpha_{21}/\Delta_1, \quad \delta_{11} = s\alpha_{11}/\Delta_1.$$

There remains a linear relation on the γ_{ij}. Since $\Delta_1 \neq 0$ requires merely that α_{11} and α_{21} shall not both vanish, *the number of operators transforming the group given by σ into that given by σ_1 is $p^3(p^2-1)$.*

COROLLARY. *The group $K_{p^2}^{**} = \{[\sigma\gamma, 0, c, \gamma]\}$, where σ is a particular not-square, is self-conjugate only under a group of order $p^3(p^2-1)$.*

18. THEOREM. *Within $G_{n=1}$, the group K_{p^2} is self-conjugate only under $G_{\frac{1}{2}p^4(p-1)^2}$. Within G for n odd, it is self-conjugate only under the group of*

operators (6) *with* α_{11} *and* α_{12} *integers. Within* G *for* n *even, it is self-conjugate only under the group of operators* (6) *with* α_{11}^2 *and* $\alpha_{11}\alpha_{12}$ *integers.*

We proceed as in § 4 with $a = \gamma = 0$. Then

$$A_2 = -k\beta_{11}^2 - 2c\beta_{11}\beta_{12}, \qquad B_2 = 1 + k\alpha_{11}\beta_{11} + c\alpha_{11}\beta_{12} + c\beta_{11}\alpha_{12},$$

$$C_2 = -k\beta_{11}\beta_{21} - c\beta_{11}\beta_{22} - c\beta_{12}\beta_{21}, \qquad D_2 = k\beta_{11}\alpha_{21} + c\beta_{11}\alpha_{22} + c\alpha_{21}\beta_{12}.$$

The condition $A_2 = 0$ for every k and c requires that $\beta_{11} = 0$. Then $B_2 = 1$ gives $\alpha_{11}\beta_{12} = 0$, $C_2 = 0$ gives $\beta_{12}\beta_{21} = 0$, $D_2 = 0$ gives $\alpha_{21}\beta_{12} = 0$. Hence $\beta_{12} = 0$. Proceeding as in § 16, we must have A, B, Γ, E all zero for every k and c. Now

$$B = -k\beta_{21}^2 - 2c\beta_{21}\beta_{22}, \quad \Gamma = k\alpha_{21}^2 + 2c\alpha_{21}\alpha_{22}, \quad A = -k\alpha_{11}\beta_{21} - c\alpha_{11}\beta_{22} - c\alpha_{12}\beta_{21}.$$

Hence $\beta_{21} = 0$, $\alpha_{21} = 0$, $\alpha_{11}\beta_{22} = 0$. If $\alpha_{11} = 0$, all the coefficients in the first column of (1) would vanish; hence $\beta_{22} = 0$. The abelian conditions now give

$$\alpha_{22}\delta_{22} = 1, \quad \delta_{12}\alpha_{22} = 0, \quad \alpha_{11}\delta_{11} + \alpha_{12}\delta_{12} = 1, \quad \alpha_{11}\delta_{21} + \alpha_{12}\delta_{22} = 0.$$

Hence $\delta_{12} = 0$, so that the transformer reduces to (6). Further,

$$E = 0, \qquad K = k\alpha_{11}^2 + 2c\alpha_{11}\alpha_{12}, \qquad C = c\alpha_{11}\alpha_{22}.$$

Hence K and C are integers for every k and c if and only if α_{11}^2 and $\alpha_{11}\alpha_{12}$ are integers. If n be odd, α_{11} must be an integer.

19. THEOREM. *Within* $G_{n=1}$ *the group* $K_{p^2}^*$ *is self-conjugate only under the group* $H_{p^2(p-1)^2}$ *of the operators*

$$(25)\quad U = \pm\begin{bmatrix} \alpha_{11} & \gamma_{11} & 0 & \gamma_{12} \\ 0 & \alpha_{11}^{-1} & 0 & 0 \\ 0 & \alpha_{11}^{-1}\alpha_{22}\gamma_{12} & \alpha_{22} & \gamma_{22} \\ 0 & 0 & 0 & \alpha_{22}^{-1} \end{bmatrix}, \quad V = \pm\begin{bmatrix} 0 & \alpha_{21}^{-1}\alpha_{12}\gamma_{22} & \alpha_{12} & \gamma_{12} \\ 0 & 0 & 0 & \alpha_{12}^{-1} \\ \alpha_{21} & \gamma_{21} & 0 & \gamma_{22} \\ 0 & \alpha_{21}^{-1} & 0 & 0 \end{bmatrix}.$$

Proceeding as in § 4 with $a = c = 0$, we find that $A_2 = -k\beta_{11}^2 - \gamma\beta_{12}^2 = 0$ for every k and γ requires that $\beta_{11} = \beta_{12} = 0$. Then $B_2 = 1$, $C_2 = 0$, $D_2 = 0$, identically. Proceeding as in § 16, with also $c = 0$, we have

$$B = -k\beta_{21}^2 - \gamma\beta_{22}^2, \qquad C = k\alpha_{11}\alpha_{21} + \gamma\alpha_{12}\alpha_{22}.$$

Then $B = 0$, $C = 0$, for every k and γ, give $\beta_{21} = \beta_{22} = 0$, $\alpha_{11}\alpha_{21} = \alpha_{12}\alpha_{22} = 0$. Then $A = 0$, $E = 0$, identically. The transformer is thus of the form (19).

If $\alpha_{11} \neq 0$, then

$$\alpha_{21} = 0, \ \alpha_{22} \neq 0, \ \alpha_{12} = 0, \ \delta_{21} = 0, \ \delta_{12} = 0, \ \alpha_{11}\delta_{11} = \alpha_{22}\delta_{22} = 1.$$

If $\alpha_{11} = 0$, then

$$\alpha_{21} \neq 0,\ \alpha_{12} \neq 0,\ \alpha_{22} = 0,\ \delta_{11} = 0,\ \delta_{22} = 0,\ \alpha_{12}\delta_{12} = \alpha_{21}\delta_{21} = 1.$$

The operators U form the group $(K_{p^3},\ T_{1,\,a_{11}}\,T_{2,\,a_{22}})$. The operators V are given uniquely by the products UP_{12} [or also by $P_{12}\,U$].

The types of non-conjugate subgroups of order p^2 in $G_{n=1}$.

20. By § 9, we may confine the discussion to the subgroups of H_{p^3}, G_{p^3} and K_{p^3}. Each of these three groups has the self-conjugate subgroup K_{p^2}. By § 11 the only additional subgroups of order p^2 in H_{p^3} are, for $p > 3$, the p non-cyclic groups $H_{p^2}^{a,\,c}$, with $a \neq 0$, and, for $p = 3$, 3 cyclic groups of order 9. The latter are all conjugate within G_{25920}, each being self-conjugate only in a group of order 27.* *Hence these cyclic groups are all conjugate and each is self-conjugate only under the group H_{27}.*

Now $[p, q, r, s]$ transforms $[k, \rho a, \delta, \rho a]$ into $[k', \rho a, c', \rho a]$, where $c' = \delta + \rho q a - \rho s a$. For the groups† (16), $a \neq 0$, $\delta = \rho c + \frac{1}{2}\rho(\rho - 1)a^2$. Taking $s = 0$, $q = -c/a$, we have $c' = \frac{1}{2}\rho(\rho - 1)a^2$. *Hence the groups $H_{p^2}^{a,\,c}$, $a \neq 0$ are all conjugate with*

$$(16')\quad H_{p^2}^{a,\,0} = \{[k, \rho a, \tfrac{1}{2}\rho(\rho - 1)a^2, \rho a], (k, \rho = 0, 1, \cdots, p - 1)\}.$$

To find the group transforming (16′) into itself, we proceed as in § 4 and find that the transformer must be of the form (7) with $\beta_{22} = 0$. The latter transforms the general operator of (16′) into $[k', \rho a\alpha_{11}\delta_{22}, c', \rho a\alpha_{22}^2]$, where

$$c' = \tfrac{1}{2}\rho(\rho - 1)a^2\alpha_{11}\alpha_{22} - \rho a\alpha_{11}\gamma_{22} + \rho a\alpha_{12}\alpha_{22}.$$

Hence must $\alpha_{11}\delta_{22} = \alpha_{22}^2$, $c' = \frac{1}{2}\rho\alpha_{22}^2(\rho\alpha_{22}^2 - 1)a^2$. Since $\alpha_{22}\delta_{22} = 1$, we get

$$\alpha_{11} = \alpha_{22}^3,\quad -\tfrac{1}{2}a\alpha_{22}^2 = -\alpha_{11}\gamma_{22} + \alpha_{12}\alpha_{22} - \tfrac{1}{2}a\alpha_{11}\alpha_{22}.$$

Hence (16′) *is self-conjugate only under a subgroup of order $\frac{1}{2}p^3(p - 1)$ of $G_{n=1}$.*

The group G_{p^3} contains in addition to K_{p^3} only the p groups (17) with $a \neq 0$. Transforming $[k, ra, rc, 0]$ by the general operator (7) of the largest subgroup in which G_{p^3} is self-conjugate, we obtain $[k', ra', rc', 0]$, where

$$a' = a\alpha_{11}\delta_{22} - c\alpha_{11}\beta_{22},\qquad c' = c\alpha_{11}\alpha_{22} - a\alpha_{11}\gamma_{22},\qquad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

Since $a \neq 0$, we can determine α_{22}, β_{22}, γ_{22}, δ_{22} so that these three conditions are satisfied, whatever be the values, not both zero, of a' and c'. Thus

$$\delta_{22} = (a' + c\alpha_{11}\beta_{22})/a\alpha_{11}.\qquad \gamma_{22} = (c\alpha_{11}\alpha_{22} - c')/a\alpha_{11}$$

*Transactions, vol. 2 (1901), p. 138. This also follows from the discussion of (16′).

† We now write ρ instead of r, to avoid confusion with the new r.

from the first two, so that the third reduces to $a'\alpha_{22} + c'\beta_{22} = a\alpha_{11}$. Each of the groups is therefore conjugate with $\{[k', 0, rc', 0], (k', r = 0, 1, \cdots, p-1)\}$. *Hence the* $p+1$ *subgroups of order* p^2 *of* G_{p^3} *are conjugate within* $G_{n=1}$.

Finally, the p^2+p+1 subgroups of order p^2 of K_{p^3} are conjugate with K_{p^2}, $K_{p^2}^*$ or $K_{p^2}^{**} = \{[\sigma\gamma, 0, c, \gamma]\}$, the latter being conjugate with neither of the first two (§ 17). Since these groups are self-conjugate only under subgroups of orders $\frac{1}{2}p^4(p-1)^2$, $p^3(p-1)^2$, $p^3(p^2-1)$, respectively (§§ 17–19), while for (16′) the order is $\frac{1}{2}p^3(p-1)$, no two of these four groups are conjugate.

THEOREM. *For* $p \geqq 3$, *there are exactly four types of non-conjugate subgroups of order* p^2 *of the group* $G_{n=1}$.

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ON THE ORDER OF LINEAR HOMOGENEOUS GROUPS*

BY

H. F. BLICHFELDT

The different types of finite groups of linear homogeneous substitutions in two and three variables have been enumerated by KLEIN, GORDAN, C. JORDAN and VALENTINER. The different types of such groups in two variables were determined by KLEIN† through geometrical considerations; GORDAN‡ made the problem depend upon that of finding the different solutions of the equation $1 + \cos\phi_1 + \cos\phi_2 + \cos\phi_3 = 0$, ϕ_1, ϕ_2, ϕ_3 being rational angles. JORDAN§ and VALENTINER‖ constructed certain fundamental equations involving the orders of the different types of groups in three variables, which equations would furnish a finite number of groups only.

JORDAN then attempted to employ his method in enumerating all the groups in four variables,¶ but found the complete discussion of his fundamental equation well nigh impossible. Even in the case of three variables the work was a formidable one, as shown by the fact that two simple groups of orders 168 and 360 respectively escaped his notice.

It may, therefore, not be amiss to give some general theorems of a simple nature bearing on the order of the linear homogeneous groups in n variables. In particular, limits are found to the number of different primes dividing the order of the "primitive groups."

The following explanation of technical terms and phrases used will be necessary. Any substitution S of a linear homogeneous group H of finite order in n variables $x_1, x_2, \cdots, x_n$ will be of the form

$$x_i' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad (i = 1, 2, \cdots, n).$$

We shall say the group is of *degree* n.

* Presented to the Society (San Francisco), April 25, 1903, under a different title. Received for publication June 15, 1903.

† Mathematische Annalen, vol. 9 (1875), p. 183.

‡ Mathematische Annalen, vol. 12 (1877), p. 28.

§ Journal für Mathematik, vol. 84 (1878), p. 89.

‖ Kjóbenhavnske Skr. (6), vol. 5 (1889), p. 61.

¶ Atti della Reale Accademia di Napoli, vol. 8 (1879).

If it is possible to choose new variables $y_1, y_2, \cdots, y_n$, where

$$y_i = b_{i1}x_1 + b_{i2}x_2 + \cdots + b_{in}x_n \qquad (i = 1, 2, \cdots, n),$$

such that all the substitutions of H are of the form

$$y_i' = \alpha_{i1}y_1 + \alpha_{i2}y_2 + \cdots + \alpha_{im}y_m \qquad (i = 1, 2, \cdots, m;\ m < n),$$

$$y_j' = \beta_{j,m+1}y_{m+1} + \beta_{j,m+2}y_{m+2} + \cdots + \beta_{j,n}y_n \quad (j = m+1, m+2, \cdots, n),$$

we shall say that the group H is *intransitive*, otherwise the group is said to be *transitive*.*

A transitive group is said to be *imprimitive*, if the variables of the group can be so selected that they fall in systems of m each, $m < n$, of such a nature that any substitution of the group will transform all the variables of any one system into linear functions of the variables of the same or another system. If the variables of a transitive group cannot be so selected, the group is said to be *primitive*.

It has been proved in several ways† that the variables may be so chosen that a substitution S of finite period can be thrown into *canonical form*:

$$x_i' = \alpha_i x_i \qquad (i = 1, 2, \cdots, n),$$

in which case the constants α_i, called the *multipliers* of the substitution, are certain roots of unity. In fact, if k is the least integer for which the equation $\alpha^k = 1$ is satisfied by $\alpha_1, \alpha_2, \cdots, \alpha_n$, then is k the order of the above substitution. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, the substitution is called a *similarity-substitution*.

Now, it may be possible so to choose the variables that all the substitutions of of a group H have the canonical form. In such an event we shall say that the group is written in *canonical form*. A group that can be written in canonical form is plainly *abelian*, i. e., its substitutions are permutable, and it is intransitive, if $n > 1$.

Since every group considered is linear and homogeneous, we shall, as a rule, dispense with the adjectives "linear" and "homogeneous" with reference to a group.

§ 1.

THEOREM I. *Every abelian group can be written in canonical form.*‡

* MASCHKE has used the word "intransitive" with the meaning given here. See Mathematische Annalen, vol. 52 (1899), p. 363.

† See MOORE, *An Universal Invariant for Finite Groups of Linear Substitutions, etc.*, Mathematische Annalen, vol. 50 (1898), p. 215, for proof and references.

‡ BURNSIDE, Proceedings London Mathematical Society (1898), p. 331; L. E. DICKSON, these Transactions (1902), pp. 292–293.

THEOREM II. *If a group H has a self-conjugate subgroup G which is abelian and whose substitutions are not all similarity-substitutions, the group H is intransitive or imprimitive.*

By a proper choice of the variables $x_1, x_2, \cdots, x_n$, the group G will be written in canonical form (Theorem I). Suppose, if G be so written, its substitutions have the form

$$x_1' = \alpha x_1,\ x_2' = \alpha x_2,\ x_3' = \beta x_3,\ x_4' = \beta x_4,\ x_5' = \gamma x_5,\ \cdots,\ x_n' = \kappa x_n,$$

where no two of the multipliers $\alpha, \beta, \gamma, \cdots, \kappa$ are equal for every substitution of G. Then are the expressions

$$a_1 x_1 + a_2 x_2,\ a_3 x_3 + a_4 x_4,\ a_5 x_5,\ \cdots,\ a_n x_n,$$

$a_1, a_2, \cdots$ being arbitrary constant, *relative invariants* of G.

Now, since G is self-conjugate in H, any substitution of H will transform the system of invariants just given into another such system, say

$$b_1 x_1 + b_2 x_2,\ b_3 x_3 + b_4 x_4,\ b_5 x_5,\ \cdots,\ b_n x_n.$$

This substitution will therefore transform the variables x_1, x_2 into linear functions of themselves or into linear functions of x_3, x_4 or any other set of two letters that constitute the variables of an invariant of G. It is readily seen that the group H is intransitive or imprimitive.

§ 2.

THEOREM III. *If the order h of a group H in n variables is $p^a q^b \cdots$, where $p, q, \cdots$ are primes each greater than n, then must H be abelian.*

The theorem is self-evident for $n = 1$. Assume the theorem true for all groups of the kind considered in less than n variables, to prove it true for any such group in n variables.

We may restrict ourselves to the case of groups whose substitutions have the determinant 1. For it is plain that if the group of substitutions of determinant 1 obtained from any group H in the manner indicated in WEBER'S *Algebra*, II, 2d edition, pp. 188–189, is abelian, the group H will be abelian.

Let us therefore consider a group G of linear substitutions of determinant 1. Let the order of G be $p^{a_1} q^{b_1} \cdots$; $p, q, \cdots$ being primes each greater than n, the degree of G. It is evident that this group can contain no similarity-substitutions except the identical substitution.

If G is non-abelian, all of its subgroups may be abelian, or it must contain at least one non-abelian subgroup all of whose subgroups are abelian. Let G' be such a non-abelian group, of order $p^{a_2} q^{b_2} \cdots$. Groups of this character have

been studied by G. A. MILLER and H. C. MORENO,* who proved that they are composite.

It follows (Theorem II) that G' is intransitive or imprimitive. In any event, it is plain that the group generated by the $n!$th powers of the substitutions of G' is intransitive. The group so generated is, however, the group G' itself, the order of this being prime to $n!$.

The groups obtained from G' by selecting its different systems of transitivity are, by assumption, abelian. This must, therefore, also be the case with G', contrary to the supposition as to the nature of G'. Thus G must be abelian, and therefore also H.

§ 3.

In the proof of the next theorem the following lemma is used:

LEMMA. *Let $E=0$ be an equation, the left-hand member of which is the sum of certain roots of unity. If the factors of the indices of these roots are powers of primes $p, q, r, \cdots$, the exponents of these powers being less than $a+1, b+1, c+1, \cdots$, respectively, then we may write each of the roots considered in the form:*

$$\bar{\alpha}^A \bar{\beta}^B \bar{\gamma}^C \cdots \alpha_1^{A_1} \beta_1^{B_1} \gamma_1^{C_1} \cdots,$$

where $\alpha_1, \beta_1, \gamma_1, \cdots$ are, respectively, primitive roots of the equations

$$\theta^p - 1 = 0,\ \theta^q - 1 = 0,\ \theta^r - 1 = 0, \cdots,$$

and where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots$, are, respectively, primitive roots of the equations $\theta^{p^a} - 1 = 0,\ \theta^{q^b} - 1 = 0,\ \theta^{r^c} - 1 = 0, \cdots$. Moreover, the numbers

$$A, B, C, \cdots, A_1, B_1, C_1, \cdots$$

may be so chosen that $A < p^{a-1}$, $A_1 < p$; $B < q^{b-1}$, $B_1 < q$; etc.

Now, let the terms of E be so written, and let us substitute the arbitrary letters $x, y, z, \cdots$ for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots$, and $x_1, y_1, z_1, \cdots$ for $\alpha_1, \beta_1, \gamma_1, \cdots$, calling the resulting expression E'. If then we replace the powers

$$y_1, y_1^2, \cdots, y_1^{q-1};\ z_1, z_1^2, \cdots, z_1^{r-1};\ \cdots$$

by the numbers $+1, 0, -1$ in any manner such that we get

$$1 + y_1 + y_1^2 + \cdots + y_1^{q-1} = 0,\ 1 + z_1 + z_1^2 + \cdots + z_1^{r-1} = 0, \cdots,$$

and if we replace $x_1, x, y, z, \cdots$ and their powers by $+1$, the resulting value of E' will be an integer $\equiv 0\ (mod.\ p)$.

* *Non-abelian Groups in which every Subgroup is Abelian*, these Transactions, vol. 4 (1903), pp. 398–404.

The first part of this lemma is evident. To prove the second part we need the following theorem due to KRONECKER *: *if p is a prime and a any integer, the expression* $X \equiv 1 + x^{p^{a-1}} + x^{2p^{a-1}} + \cdots + x^{(p-1)p^{a-1}}$ *can not be decomposed into factors of lower degree in* x, *whose coefficients are rational functions of a primitive root of unity, unless the exponent of this root is divisible by* p.

It follows from this theorem that if the coefficients c_i in the function $f(x) \equiv \sum c_i x^i$ are sums of roots of unity, whose indices are prime to p, and if $f(\alpha) = 0$, where α is a primitive root of the equation $\theta^{p^a} - 1 = 0$, then is $f(x)$ divisible by X. Moreover, we readily see that if we write $f = f_1 + f_2 + f_3 + \cdots$, where f_k consists of all of the terms of $\sum c_i x^i$ for which the exponents i will give the same remainder $r_k \pmod{p^{a-1}}$, then must every expression f_k be divisible by X. †

It follows that if the coefficients $c_{i,j}$ of the function

$$f(x, x_1) \equiv \sum_{i=0}^{p^{a-1}-1} \sum_{j=0}^{p-1} c_{i,j} x^i x_1^j$$

are sums of roots of unity whose indices are prime to p, and if $f(\bar{\alpha}, \alpha_1) = 0$, where $\bar{\alpha}$ and α_1 are, respectively, primitive roots of $\theta^{p^a} - 1 = 0$, $\theta^p - 1 = 0$, then is $f(x, x_1)$ divisible by $1 + x_1 + x_1^2 + \cdots + x_1^{p-1}$, and the quotient is free from x_1. From this again it follows that we can write the expression E' of the lemma in the form

$$E' = (1 + x_1 + x_1^2 + \cdots + x_1^{p-1}) E_1 + (1 + y_1 + y_1^2 + \cdots + y_1^{q-1}) E_2 + (1 + z_1 + z_1^2 + \cdots + z_1^{r-1}) E_3 + \cdots,$$

where the expressions E_i are integral functions of $x, y, z, \cdots, x_1, y_1, z_1, \cdots$, with integral coefficients. In addition, E_1 is free from x_1, E_2 from y_1, E_3 from z_1, etc. The truth of the second part of the lemma is seen immediately.

§ 4.

THEOREM IV. *In any group H of degree n, all the substitutions whose orders are divisible by no prime less than $(n-1)(2n+1)+1$ form a subgroup.*

In any substitution T of finite period, say

$$(1) \qquad x_i' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad (i = 1, 2, \cdots, n),$$

the sum of the multipliers (which, as remarked above, are certain roots of unity),

* *Mémoire sur les facteurs irréductibles de l'expression* $x^n - 1$, Journal de Mathématiques pures et appliquées, t. 19 (1854), p. 178.

† This argument is given by GORDAN: *Ueber endliche Gruppen*, etc., Mathematische Annalen, vol. 12 (1877), pp. 29–30.

is equal to $a_{11} + a_{22} + \cdots + a_{nn}$, as they are the roots of the *characteristic equation* of T.* We shall call this sum the *weight* of the substitution T.

Now, let S and T be any two substitutions of a group H in n variables, and let the orders of S and T be p^α and $p^a q^b \cdots$ respectively, $p, q, \cdots$ being primes each greater than $(n-1)(2n+1)$.

Let the variables be chosen in such a manner that S has the canonical form. Suppose this to be

$$x_1' = \alpha x_1,\ x_2' = \alpha x_2,\ x_3' = \beta x_3,\ x_4' = \beta x_4,\ x_5' = \gamma x_5,\ \cdots,\ x_n' = \kappa x_n,$$

where $\alpha \neq \beta \neq \gamma \neq \cdots \neq \kappa$. Let T be of the form (1).

Then if the substitutions

$$T,\ ST,\ S^2T,\ \cdots,\ S^{p^\alpha-1}T,$$

be formed, we will find certain linear relations among their weights.* Indicating these by $(T), (ST), \cdots, (S^{p^\alpha-1}T)$, we have namely,

$$(2)\qquad \begin{cases} (T) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}, \\ (ST) = \alpha a_{11} + \alpha a_{22} + \beta a_{33} + \cdots + \kappa a_{nn}, \\ (S^2T) = \alpha^2 a_{11} + \alpha^2 a_{22} + \beta^2 a_{33} + \cdots + \kappa^2 a_{nn}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (S^{p^\alpha-1}T) = \alpha^{p^\alpha-1} a_{11} + \alpha^{p^\alpha-1} a_{22} + \beta^{p^\alpha-1} a_{33} + \cdots + \kappa^{p^\alpha-1} a_{nn}. \end{cases}$$

Supposing there are just m different multipliers, $\alpha, \beta, \gamma, \kappa$, in S, we can eliminate the quantities $a_{11} + a_{22}, a_{33} + a_{44}, \cdots, a_{nn}$ between $m+1$ of the equations (2), say the first m and the $r+1$th, $m \leqq r \leqq p^\alpha - 1$.

The equation thus obtained will be of the form

$$(3)\qquad \begin{vmatrix} (T) & 1 & 1 & 1 & \cdots & 1 \\ (ST) & \alpha & \beta & \gamma & \cdots & \kappa \\ (S^2T) & \alpha^2 & \beta^2 & \gamma^2 & \cdots & \kappa^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (S^{m-1}T) & \alpha^{m-1} & \beta^{m-1} & \gamma^{m-1} & \cdots & \kappa^{m-1} \\ (S^rT) & \alpha^r & \beta^r & \gamma^r & \cdots & \kappa^r \end{vmatrix} \equiv (T)A_0' - (ST)A_1' + \cdots + (-1)^m (S^rT)A_m' = 0,$$

say, which equation, it may be noticed, is true for all values of r.

*JORDAN, Journal für Mathematik, vol. 84 (1878), p. 112.

†Compare GORDAN, Mathematische Annalen, vol. 12 (1877), pp. 23-25; MASCHKE, ibid., vol. 50 (1898), p. 492.

As $\alpha, \beta, \gamma, \cdots$ are all different and are p^sth roots of unity, the alternant $\sum \pm \alpha\beta^2 \cdots \kappa^{m-1}$ is not zero and is plainly a factor of each of the coefficients $A_0', A_1', \cdots$. Dividing this factor out, we obtain a typical form of the desired linear relations

$$E = (T)A_0 - (ST)A_1 + \cdots + (-1)^{m-1}(S^{m-1}T)A_{m-1} + (-1)^m(S^rT) = 0. \tag{4}$$

§ 5.

We can apply the lemma of § 3 to the equation (4). To begin with, let 1 be substituted for every root of the equation $(\theta^{p^r}-1)(\theta^{p^s}-1)=0$ that may occur in (4), and let $y, z, \cdots, y_1, z_1, \cdots$ be written for all other roots, according to the rule laid down in the lemma. If then we substitute 1 for $y, z, \cdots$ and their powers, and assign to the powers $y_1, y_1^2, \cdots, y_1^{q-1}, z_1, z_1^2 \cdots$, etc., the values -1, 0 or $+1$ in any manner such that we get

$$1 + y_1 + y_1^2 + \cdots + y_1^{q-1} = 0, \quad \text{etc.},$$

the left-hand member of (4) becomes an integer $\equiv 0 \pmod{p}$. Let the result be written in the form

$$E'' \equiv [T]B_0 - [ST]B_1 + \cdots + (-1)^{m-1}[S^{m-1}T]B_{m-1} + (-1)^m[S^rT] \equiv 0 \pmod{p}, \tag{5}$$

where B_k is the resulting value of A_k, which contained no other root than those of $\theta^{p^r}-1=0$ and $\theta^2-1=0$; and $[S^kT]$ that of (S^kT).

As each of the weights (S^kT) is the sum of n roots of unity, it is easily seen that the expressions $[S^kT]$ must be integers lying between $-n$ and $+n$ inclusive.

It remains for us to find the number B_k, the resulting value of the coefficient A_k, which is the ratio of two minors of the determinant (3). In the quotient, 1 is to be substituted for the quantities $\alpha, \beta, \cdots, \kappa$, which are roots of the equation $\theta^{p^r}-1=0$.

JACOBI has given the value of the determinant of m^2 elements

$$\begin{vmatrix} \alpha^a & \beta^a & \cdots & \kappa^a \\ \alpha^b & \beta^b & \cdots & \kappa^b \\ \alpha^c & \beta^c & \cdots & \kappa^c \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

in the form

$$\begin{vmatrix} {}_mH_a & {}_mH_{a-1} & \cdots & {}_mH_{a-m+1} \\ {}_mH_b & {}_mH_{b-1} & \cdots & {}_mH_{b-m+1} \\ {}_mH_c & {}_mH_{c-1} & \cdots & {}_mH_{c-m+1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha & \beta & \cdots & \kappa \\ \alpha^2 & \beta^2 & \cdots & \kappa^2 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where ${}_mH_t = 0$, if $t < 0$; ${}_mH_0 = 1$; and where ${}_mH_t$, $t \geqq 1$, represents the sum of all products of degree t in the m letters $\alpha, \beta, \cdots, \kappa$. Substituting 1 for each of these letters in ${}_mH_t$, $t \geqq 1$, this becomes

$$\frac{(m+t-1)!}{(m-1)!\,t!} = {}_mH'_t,$$

say.

Now we easily find

$$\begin{vmatrix} {}_mH'_a & {}_mH'_{a-1} & \cdots & {}_mH'_{a-m+1} \\ {}_mH'_b & {}_mH'_{b-1} & \cdots & {}_mH'_{b-m+1} \\ {}_mH'_c & {}_mH'_{c-1} & \cdots & {}_mH'_{c-m+1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} {}_1H'_a & {}_2H'_{a-1} & \cdots & {}_mH'_{a-m+1} \\ {}_1H'_b & {}_2H'_{b-1} & \cdots & {}_mH'_{b-m+1} \\ {}_1H'_c & {}_2H'_{c-1} & \cdots & {}_mH'_{c-m+1} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & \cdots & a^{m-1} \\ 1 & b & b^2 & \cdots & b^{m-1} \\ 1 & c & c^2 & \cdots & c^{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \div \{2!\,3!\cdots(m-1)!\}.$$

Hence,

$$B_0 = \begin{vmatrix} 1 & 1 & 1 & \cdots \\ 1 & 2 & 2^2 & \cdots \\ 1 & 3 & 3^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & m-1 & (m-1)^2 & \cdots \\ 1 & r & r^2 & \cdots \end{vmatrix}$$

$$\div \{2!\,3!\cdots(m-1)!\} = \frac{(r-1)(r-2)\cdots(r-m+1)}{(m-1)!};$$

$$B_1 = \frac{r(r-2)(r-3)\cdots(r-m+1)}{(m-2)!}, \cdots,$$

$$B_k = \frac{r(r-1)(r-2)\cdots(r-m+1)}{(r-k)(m-k-1)!\,k!} \qquad (k \leqq m-1).$$

Substituting in (4) and multiplying by $(p-1)!$ $(p > m-1)$ we have

$$(6) \quad P_r = \{[T]B_0 - [ST]B_1 + \cdots + (-1)^{m-1}[S^{m-1}T]\}(p-1)!$$
$$\equiv (-1)^m [S^r T] \pmod{p},$$

the expression P_r being an integral function of r with integral coefficients and of degree $m-1$ or less. If the coefficients of $r, r^2, \cdots, r^{m-1}$ are $\equiv 0 \pmod{p}$, we must have $P_0 \equiv P_1 \equiv \cdots \equiv P_{m-1} \pmod{p}$, or

$$(-1)^{m-1}[T] \equiv (-1)^{m-1}[ST] \equiv \cdots \equiv (-1)^{m-1}[S^{m-1}T] \pmod{p}.$$

Accordingly, if $[T] \not\equiv [ST] \pmod{p}$, P_r must be an integral function of r of the first degree at least. Assuming this to be the case, let us substitute in P_r of (6) the values $0, 1, 2, \cdots, p-1$ for r. This takes a number of different values, the remainders of which should all lie between the limits $-n$ and $+n$ inclusive, in order that the congruences (6) may be satisfied. Now, by a well known theorem, the number of different values $\pmod{p}$ of r which, when substituted in P_r, will give a definite remainder $\pmod{p}$, can not exceed the degree of P_r in r. Accordingly, the total number of different values $\pmod{p}$ of r that could satisfy the congruence (6) can not exceed $(m-1)(2n+1)$. Therefore, as $p > (n-1)(2n+1) \geqq (m-1)(2n+1)$, the congruences (6) can not be satisfied for all values r under the assumption $[T] \not\equiv [ST] \pmod{p}$. Thus $[T] \equiv [ST] \pmod{p}$, and as $|[T]| + |[ST]| \leqq 2n < p$, we must have $[T] = [ST]$. Hence, if p *be any prime greater than* $(n-1)(2n+1)$, *the congruences* (6) *require* $[T] = [ST]$.

It follows from this that (ST) can not contain roots of unity having prime indices different from those of the roots in (T) and (S). For, since the prime indices $p, q, \cdots$ of the roots contained in (T) and (S) are each greater than $2n+1$, we could otherwise distribute the numbers $-1, 0, +1$ in such a manner among the roots designated above by $y_1, z_1, \cdots, y_1^2, z_1^2, \cdots$, etc., that the value of $[T]$ becomes n, and so that the value of $[ST]$ becomes less than n numerically.

Accordingly, if the order of T is the product of powers of primes each greater than $(n-1)(2n+1)$, and the order of S is a power of a prime p greater than $(n-1)(2n+1)$, the order of ST must be a product of powers of primes each greater than $(n-1)(2n+1)$. Then, by a well-known theorem concerning the resolution of any substitution into a product of substitutions of the form S, we see the truth of Theorem IV.

It may be remarked that the limit considered in this theorem, $(n-1)(2n+1)$, is, in general, higher than necessary (see §6). The analysis just given leads to this question: what is the smallest number $\tau_n \leqq (n-1)(2n+1)$ such that no function exists of the form $P_r = ar^{n-1} + br^{n-2} + \cdots + k$ of degree not less than 1 and with integral coefficients, whose remainders $\pmod{p}$, p being any given prime greater than τ_n, will not all lie between $-n$ and $+n$ inclusive? The answer given above, $\tau_n = (n-1)(2n+1)$ is that most readily found, but is, as remarked, unnecessarily high in general.

From Theorems III and IV it follows that all the substitutions whose orders are products of powers of primes each greater than $(n-1)(2n+1)$ form an abelian subgroup of the group H, which is evidently self-conjugate in H.

§ 6.

We shall now consider more particularly the groups of degrees 3, 4, 5 and 6. We shall restrict ourselves to groups whose substitutions are of determinant 1, in which case none of the substitutions except identity considered in Theorem IV can be similarity-substitutions. We can therefore employ Theorem II directly.

Concerning the groups in three variables.

As remarked above, the limit $(n-1)(2n+1)$ can, in general be reduced. For $n=3$ the reduction depends upon the solution of the following problem: what is the least number τ_3 such that for any prime $p > \tau_3$ no function exists of the form $ar^2+br+c \equiv c \pmod{p}$, a, b, c being integers, all of whose remainders (mod. p) lie between -3 and $+3$ inclusive? By trying in turn all the primes less than 14 $[=(n-1)(2n+1)]$ we find that τ_3 must be 7 at least, but need not be higher. Accordingly, we have the

THEOREM V. *The order of a primitive group in three variables has the prime factors* 2, 3, 5 *and* 7 *only.*

Concerning the groups in four and five variables.

Here we have to find if functions

$$ar^3+br^2+cr+d \equiv d,\ ar^4+br^3+cr^2+dr+e \equiv e \pmod{p},$$

exist, all of whose remainders lie between -4 and $+4$, -5 and $+5$ (limits included), respectively. In the first case, no such function exists if p is a prime greater than 13, and no function of the second form exists for a prime p greater than 19. Hence

THEOREM IV. *The order of a primitive group in four variables is divisible by no prime greater than* 13. *The order of a primitive group in five variables is divisible by no prime greater than* 19.

Concerning groups in six variables.

The limit is here found to be 23.* An exception occurs in the case of $p=31$. The function r^5 has the remainders (mod. 31) $0, \pm 1, \pm 5, \pm 6$. By a more detailed study of the possible forms of the quantities $[S^kT]$ (§ 5)

* The limit may be 19 for $n=6$. The author has not considered whether functions may exist of the form $ar^5+br^4+cr^3+dr^2+er+f$ all of whose remainders (mod. 23) lie between -6 and $+6$ inclusive.

it is found, however, that if one of these numbers can be $+5$, another may be put equal to 4 or 3 by a proper distribution of the numbers $-1, 0, +1$ among the roots indicated in § 5 by $y_1, z_1, \cdots, y_1^2, z_1^2, \cdots$, etc. As the remainders of r^5 ought to be the numbers $[S^k T]$, we may exclude the prime $p = 31$.

THEOREM VII. *The order of a primitive group in six variables is divisible by no prime greater than* 23.

The orders of imprimitive and intransitive groups do not, of course, obey these laws. But such groups are, in general, of a simpler type. Thus, the imprimitive groups in three variables are of the type

$$x_1' = ax_i, \quad x_2' = bx_j, \quad x_3' = cx_k, \qquad (i, j, k = 1, 2, 3);$$

the constants a, b, c being certain roots of unity and different for the different substitutions.

Again, either an imprimitive group in four variables has the type

$$x_1' = ax_i, \; x_2' = bx_j, \; x_3' = cx_k, \; x_4' = dx_l, \quad (i, j, k, l = 1, 2, 3, 4);$$

or else its substitutions can be divided into two sets of the following types:

$$x_1' = ax_3 + bx_4, \quad x_2' = cx_3 + dx_4, \quad x_3' = \alpha x_1 + \beta x_2, \quad x_4' = \gamma x_1 + \delta x_2;$$

$$x_1' = a'x_1 + b'x_2, \quad x_2' = c'x_1 + d'x_2, \quad x_3' - \alpha' x_3 + \beta' x_4, \quad x_4' = \gamma' x_3 + \delta' x_4.$$

The construction of such a group would depend upon the construction of the primitive groups in two variables.

In the same way the construction of the imprimitive groups in six variables would depend upon the construction of the primitive groups in two and three variables.

STANFORD UNIVERSITY,
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NON-ABELIAN GROUPS

IN WHICH EVERY SUBGROUP IS ABELIAN*

BY

G. A. MILLER AND H. C. MORENO

Several years ago DEDEKIND and others investigated the groups in which every subgroup is invariant, and found that the theory of these groups presents remarkably few difficulties except such as are involved in abelian groups. The non-abelian groups in which every subgroup is abelian present a parallel example of simple and general results. The following are some of the most important ones: All such groups are solvable. Their orders cannot be divided by more than two distinct primes. Every commutator is of prime order. When the order is $p^\alpha q^\beta$, (p and q being prime; $\alpha, \beta > 0$), there are just q^β subgroups of order p^α and there is only one subgroup of order q^β. The former are cyclic and the latter is of type $(1, 1, 1, \cdots)$. When the order is p^α, there are just $p + 1$ subgroups of order $p^{\alpha-1}$ and none of them involves more than three invariants. If there are three invariants at least one of them must be of order p.

Let G represent any non-abelian group in which every subgroup is abelian. We shall first prove that G is solvable. If G is represented as a transitive substitution group it will be either primitive or imprimitive. In the latter case it will be isomorphic with some primitive group P.† The subgroup of G which corresponds to identity in P is abelian and every subgroup of P is abelian. The group G is solvable whenever P is solvable. Hence it remains to prove that a non-abelian primitive group P in which every subgroup is abelian is always solvable. Let P_1 be the subgroup of P which is composed of all the substitutions omitting a given letter. Since P is non-regular,‡ P_1 includes at least one substitution besides the identity. If two conjugates of P_1 had a common substitution besides the identity, this substitution would be invariant under P, since P_1 is a maximal subgroup of P. Hence P must be of class $n - 1$, n being the degree of P. Therefore P contains an invariant subgroup of order n,§ with respect to which the quotient group is simply isomorphic with P_1. As

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† JORDAN, *Traité des substitutions* (1870), p. 399.

‡ JORDAN, *Traité des substitutions* (1870), p. 60.

§ FROBENIUS, Berliner Sitzungsberichte (1902), p. 455.

both this invariant subgroup and P_1 are abelian, P is solvable and hence G *is solvable.*

It will be convenient to consider all the groups in question under two headings according as the order (g) of G is divisible by more than one prime, or is a power of a single prime.

§ 1. *The order of G is divisible by more than one prime number.*

Let $g = p^\alpha q^\beta r^\gamma \cdots$ (p, q, r, $\cdots$ being distinct primes). Since G is solvable it must contain a subgroup G_1 of prime index p. The order G_1 is $g_1 = p^{\alpha-1} q^\beta r^\gamma \cdots$. Since G_1 is abelian, it is the direct product of the subgroups $P_{\alpha-1}$, Q_β, R_γ, $\cdots$, of orders $p^{\alpha-1}$, q^β, r^γ, $\cdots$ respectively. The orders of each of the operators of G which are not in G_1 is divisible by p, and hence G contains only one subgroup of each of the orders q^β, r^γ, $\cdots$. If it contained only one subgroup P_α of order p^α, it would be the direct product of the abelian groups, P_α, Q_β, R_γ. As this is impossible, G *contains only one subgroup of each of the orders* q^β, r^γ, $\cdots$, *but it contains more than one subgroup of order* p^α.

Any two of the abelian subgroups of order p^α must generate G and hence each of the operators of $P_{\alpha-1}$ is invariant under G. Let s be any operator of P_α which is not contained in $P_{\alpha-1}$. If s were commutative with each operator in the subgroups Q_β, R_γ, $\cdots$, then G would be abelian. Hence s transforms one of these subgroups, Q_β say, into itself without being commutative with each of its operators. As $\{Q_\beta, s\}$ is non-abelian it follows that g *cannot be divisible by more than two distinct primes.* We may therefore assume that $g = p^\alpha q^\beta$, and that G contains only one subgroup Q_β of order q^β, but that it contains more than one subgroup of order p^α.

Since $\{Q_\beta, s\}$ is non-abelian it is identical with G. The order of $\{Q_\beta, s\}$ is the product of q^β and the order of s. Hence s must generate P_α; i. e., P_α *is cyclic.*

We shall next prove that Q_β is of type $(1, 1, 1, \cdots)$. Let t_1 be any operator of highest order in Q_β and suppose that $t_1^q \neq 1$. The operator s is commutative with all the operators of Q_β whose orders are less than the order of t_1, otherwise these operators and s would generate a non-abelian group which would not include t_1. Let t be any operator of Q_β such that $t^q = t_1^q$. Then $t = t' t_1$, where t' is some operator of order q in Q_β. From this it follows that t has q^n values, n being the number of invariants of Q_β. As s transforms these q^n operators among themselves, it must transform at least one of them into itself. It also transforms into themselves each of the operators represented by t'. Hence s would be commutative with each operator of highest order in Q_β, if this order were supposed to exceed q; i. e., Q_β *contains no operator whose order exceeds* q.

When Q_β is cyclic G is of order $p^\alpha q$. Since each of the operators of $P_{\alpha-1}$ is invariant under G it follows that the group of cogredient isomorphisms of G is non-abelian and of order pq. Hence $q-1$ must be divisible by p. Moreover, when this condition is satisfied, we can construct one G for every value of α by establishing a $(p^{\alpha-1}, q)$ isomorphism between the cyclic group of order p^α and the non-abelian group of order pq. Since every possible G of order $p^\alpha q$ is simply isomorphic with one of these groups it follows that *there is one and only one G of order $p^\alpha q$ for every value of α, whenever $q-1$ is divisible by p. When this condition is not satisfied there is no such group.* Each of these groups contains $p^{\alpha-1}q(p-1)$ operators of order p^α in addition to the cyclic invariant subgroup of order $p^{\alpha-1}q$.

It remains to consider the case in which Q_β is non-cyclic. In this case Q_β cannot contain just q operators which are invariant under G. For, as s cannot transform any subgroup of Q_β into itself without transforming each of its operators into itself, the number of subgroups of order q in Q_β would have to be $lp+1$. Since the number of subgroups of order $q^{\beta-1}$ would also have to be $lp+1$*, s would transform at least one of these $(Q_{\beta-1})$ into itself. If $\beta>2$, $Q_{\beta-1}$ would contain operators which are not commutative with s, and if $\beta=2$ the number of subgroups of order $q^{\beta-1}$ could not be of the form $lp+1$. As each of these results is impossible, it follows that Q_β cannot contain just q operators which are invariant under G.

Suppose that Q_β contains just q^e invariant operators, $1<e<\beta$. With respect to q^{e-1} of these G would be isomorphic with a group containing at least $q-1$ invariant operators of order q. If it contained more invariant operators of this order, G would contain a non-abelian subgroup. As it has been proved above that all the subgroups of this quotient group (which contains just $q-1$ invariants operators of order q) cannot be abelian, the proof is complete. Hence $P_{\alpha-1}$ is composed of all the invariant operators of G.

The group of cogredient isomorphisms (I_0) of G is therefore non-abelian and of order pq^β, $\beta>1$. Since Q_β is abelian, contains no invariant operators or invariant subgroup, and leads to an abelian quotient group, it must be the commutator subgroup of G and each of its operators must be a commutator.† From these facts it follows that G contains just q^β cyclic subgroups of order p^α. In other words, all the operators of G which are not contained in G_1 are of order p^α, while G_1 is the direct product of Q_β and $P_{\alpha-1}$.

The group G can therefore be constructed only when $q^\beta \equiv 1 \bmod p$ and when Q_β is generated by any one of its sets of p conjugate operators. As the continued product of these p operators must be invariant and hence equal to the identity, β must be less than p. From FERMAT's theorem it follows that β is

* Cf. BURNSIDE, *Theory of Groups of Finite Order* (1897), p. 60.

† Bulletin of the American Mathematical Society, vol. 6 (1900), p. 337.

a divisor of $p-1$. Moreover G can be constructed only when I_0 can be constructed. By establishing a $(q^\beta, p^{\alpha-1})$ isomorphism between I_0 and a cyclic group of order p^α we obtain one and only one G for every value of $\alpha > 0$. As each possible G is isomorphic with such a group the construction of these groups is reduced to the construction of such groups as I_0.

Since each one of the subgroups of order p in I_0 is maximal and non-invariant, I_0 may be represented as a primitive substitution group (S) of degree q^β. Conversely every primitive group of degree q^β and order pq^β is such an I_0. The necessary and sufficient condition that S is doubly transitive is that $q^\beta - 1 = p$. In this case $q = 2$ and I_0 can always be constructed and hence the infinite system which depends upon it can also be constructed. The alternating group of degree 4 and G^{α}_{56} are the first two instances of this kind. In all other cases S is simply transitive and it will be proved that it can be constructed for one and only one value of β when p and q are given. The first two instances of this kind are G^{16}_{80} and G^{25}_{75}.

It is easy to prove that I_0 cannot be constructed when $q \equiv 1 \bmod p$. The number of subgroups of every order in Q_β is evidently $\equiv 0 \bmod p$. The number of subgroups of order q is $q^{\beta-1} + q^{\beta-2} + \cdots + q + 1$, and we may write this number $(q^{\beta-1} - 1) + (q^{\beta-2} - 1) + \cdots + (q - 1) + \beta \not\equiv 0 \bmod p$, since $1 < \beta < p$. That is, G can be constructed only for $\beta = 1$ when $q \equiv 1 \bmod p$, and there is only one group when α, p and q are given.

That I_0 can be constructed for only one value of β when $q \not\equiv 1 \bmod p$ may be proved as follows: Let β_0 be the smallest value of x such that $q^x \equiv 1 \bmod p$, and let $\beta < p$ be any other possible value of x. It is well known that $\beta = l\beta_0$, l being an integer. If p^m is the highest power of p which divides the order of the group of isomorphisms (I_0') of Q_{β_0}, we shall prove that p^{ml} is the highest power of p which divides the order of the group of isomorphisms (I') of Q_β. From FERMAT's theorem and from the condition $q \not\equiv 1 \bmod p$, it follows that

$$q^{\beta_0-1} + q^{\beta_0-2} + \cdots + q + 1 \equiv q^{\beta-1} + q^{\beta-2} + \cdots + q + 1 \equiv 0 \bmod p.$$

As $\beta = l\beta_0$ the second member may be written

$$[1 + q^{\beta_0} + \cdots + q^{(l-1)\beta_0}] \cdot [1 + q + \cdots + q^{\beta_0-2} + q^{\beta_0-1}].$$

To see that the first factor is not divisible by p, it may be written in the form $l + (q^{\beta_0} - 1) + \cdots + (q^{(l-1)\beta_0} - 1)$, where each term except the first is divisible by p. Let (I'') be a subgroup of I' which transforms each of just q^r operators of Q_β into itself. The order of I'' is $q^{r(\beta-r)}$ times the order of the group of isomorphisms of the quotient group of Q_β with respect to these invariant operators. By letting $r = (l-1)\beta_0, (l-2)\beta_0, \cdots, \beta_0$, in succession, it is easy to see that p^{ml} is the highest power of p which divides the order of I'. Hence I' contains a subgroup P_{ml} of order p^{ml} and it remains to prove that

each operator of this subgroup transforms into itself at least one subgroup of order q^{β_0} in Q_β. It is clear that P_{ml} contains a subgroup of order $p^{m(l-1)}$ which transforms each of just q^{β_0} operators of Q_β into itself. As there is a substitution of order p^m which permutes these $q^{\beta_0}-1$ operators among themselves,* the proof is complete. Hence *there is always one and only one value of β such that there exists a primitive group of order pq^β and degree q^β. When $q^\beta - 1 > p$, this primitive group is simply transitive.*

§2. *The order of G is a power of a prime.*

We shall first construct two triply infinite systems of such groups and then prove that every possible group (with the exception of a few known ones) is simply isomorphic with a group in one of these systems. Let (P_m) represent any non-abelian group of order p^m, $m > 2$, which contains an operator (t) of order p^{m-1} but no non-abelian subgroup of this order.† It is well known that there is just one such group for every value of m except when $p = 2$ and $m = 3$. In this special case there are two groups: viz., the quaternion and the octic groups. In every case the commutator subgroup is generated by an operator (t_0) of order p.

Consider the groups obtained by establishing an isomorphism between P_m, with respect to the group generated by t, and the cyclic groups of order p^{m_1}, $m_1 > 2$. All the subgroups of every one of these groups are abelian, and in each of them t_0 is a power of operators of order p^{m-1} but not of an operator of higher order. Hence no two of the triply infinite systems of groups, obtained by assigning to p, m and m_1 all possible values can be simply isomorphic, with the possible exception of those in which P_m is the octic or the quaternion group. All the groups obtained when P_m is the octic group are evidently simply isomorphic with those obtained when P_m is the quaternion. Hence we need to use only one of these groups for P_m so that we consider only one group for every value of m and p. If this is done, all the groups obtained in this manner are distinct.

The second triply infinite system may be constructed in a somewhat similar manner. Let t' and t_0 represent two independent operators of orders p^{m-2} and p, respectively $(m > 3)$ and let s_0 be an operator of order p which satisfies the conditions

$$s_0^{-1} t_0 s_0 = t_0, \qquad s_0^{-1} t' s_0 = t_0 t'_0.$$

The group $\{t', t_0, s_0\}$ is of order p^m and each of its subgroups is abelian since all the subgroups of order p^{m-1} must include $\{t_0, t'^p\}$. By establishing an isomorphism between the cyclic group of order $p^{m'}$, $m' > m-3$, and $\{t', t_0, s_0\}$

* Cf. MATHIEU, Journal de Liouville (1861), p. 241.

† All the subgroups of such a group must be abelian when $p > 2$.

with respect to $\{t_0, t'\}$, we clearly obtain a triply infinite system of non-abelian groups in which every subgroup is abelian by assigning to p, m and m' all possible values. No two of these can be simply isomorphic since t' is a non-invariant operator of lowest order in the simply infinite system for which m and p have given values. No group of this triply infinite system can be simply isomorphic with a group of the triply infinite system considered in the preceding paragraph, because every subgroup of order $p^{m+m'-2}$ in the present system includes $\{t'^p, t_0, s_1^p\}$, where s_1 involves a constituent of the highest order from the cyclic group of order of p, and hence it has three invariants, while there are just two invariants in some subgroup of index p in the preceding system.

Having proved the existence of two triply infinite systems of non-abelian groups in which every subgroup is abelian, and also that no two groups contained in these two systems can be simply isomorphic, it remains to show that (with the exception of a few known ones) every possible group (G) is simply isomorphic with a group in one of these systems. The order of G will be represented by p^α. It will first be proved that G contains just $p+1$ subgroups of order $p^{\alpha-1}$ and that such a subgroup cannot involve more than three invariants.

Let G_1 be any subgroup of order $p^{\alpha-1}$ and let s be one of the smallest operators of G which are not in G_1. Since the order of s is a power of p it must transform some invariant subgroup of order p^β $(\beta = 1, 2, \cdots \alpha - 2)$ in G_1 into itself. Moreover, it transforms the operators of one of these invariant subgroups into themselves multiplied by just p invariant operators.* As this subgroup and s must generate G, it follows that *the commutator subgroup of G is of order p*. Hence the group of cogredient isomorphisms of G is of order p^2. Since a group of cogredient isomorphisms cannot be cyclic it follows that the pth power of every operator in G is invariant.

As every subgroup of order $p^{\alpha-1}$ is abelian it must include all the invariant operators of G. Hence *G contains just $p+1$ subgroups of order $p^{\alpha-1}$*. Let t be any non-invariant operator of lowest order in G_1 and let t_0 be one of the $p-1$ commutators of order p. Every subgroup of G which includes t_0 is invariant. Since $\{t, s^p, t_0\}$ and s generate G, it is clear that $\{t, s^p, t_0\} \equiv G_1$. Hence *a subgroup of order $p^{\alpha-1}$ contains at most three invariants*. If there are three, at least one of them is of order p and t_0 may be taken as one of its independent generators.

In case G_1 is cyclic the possible groups are well known and have been noted above. When G_1 has just two invariants it may be assumed that the order of each of them exceeds p, for the groups of order p^m containing an abelian subgroup of order p^{m-1} and type $(m-2, 1)$ are known.* We proceed to consider the other possible groups, when G_1 has just two invariants. From the

*Annals of Mathematics, vol. 3 (1902), p. 180.

†Transactions of the American Mathematical Society, vol. 3 (1902), p. 383.

way in which s and t have been selected, and from the fact that the order of each of the invariants of G_1 exceeds p, it follows that s and t have only identity in common. Hence $G_1 \equiv \{t, s^p\}$. Since t_0 is in G_1, it must be of the form $t_0 = t^m s^{xp}$. Hence we may suppose that t and s were so selected that t_0 is a power of one of them.

If t_0 is a power of t it is clear that G is simply isomorphic with one of the groups in the first triply infinite system mentioned above. If t_0 is a power of s we may select a group of order p^{m-1} so as to include s. Since this is abelian it will not include t, and G must again be simply isomorphic with a group of the first system. It remains to prove that every possible group when G_1 has three invariants is simply isomorphic with a group of the second triply infinite system.

We shall hereafter suppose that G_1 has three invariants and includes a non-invariant operater t of lowest order, so that the order of s is equal to or greater than the order of t. Since t and s are independent, it may be assumed that t, s^p, t_0 are the three independent generators of G_1. Hence each of these groups is simply isomorphic with one of the groups of the second triply infinite system. We can evidently establish such a simple isomorphism by choosing the order of t' equal to that of t and the order of the isomorphic cyclic group equal to that of s, and letting t' correspond to t, t_0 to t_0 and a generator of the isomorphic cyclic group to s. Every non-abelian group of order p^α, in which every subgroup is abelian is therefore contained in one or the other of the two triply infinite systems, or else among the groups which contain either an operator of order $p^{\alpha-1}$ or an abelian group of type $(\alpha - 2, 1)$.

STANFORD UNIVERSITY,
April, 1903.

ON NILPOTENT ALGEBRAS*

BY

JAMES BYRNIE SHAW

PART I. INTRODUCTION.

BENJAMIN PEIRCE first pointed out† that with respect to any idempotent basis i of an algebra all its numbers may be divided into four classes (11), (12), (21), (22) characterized by the following multiplicative relations:

$$i(11) = (11) = (11)i;\ i(12) = (12),\ (12)i = 0;$$

$$i(22) = 0 = (22)i;\qquad i(21) = 0,\ (21)i = (21);$$

Then these classes give when multiplied into each other the results shown in the following table:

	11	12	21	22
11	11	12	0	0
12	0	0	11	12
21	21	22	0	0
22	0	0	21	22

that is, $(g'h')(g''h'')$ is $(g'h'')$ or 0 according as h' is or is not the same as g''. Each class forms a sub-algebra of the algebra, the class (11) containing the idempotent basis. The class (22) may or may not contain an idempotent. If (11) contains two distinct idempotents, the process may be repeated, giving classes which we may represent by (11), (12), (13), (21), (22), (23), (31), (32), (33), with the analogous multiplication table. This division can be carried on in the same manner, giving multiplication tables of the form of perfect quadrates. The process stops when all independent idempotents have been found and isolated, so that every class on the main diagonal contains at most one idempotent, and otherwise only nilpotents. The classes not appearing on the main diagonal contain *skew* numbers only, and their squares vanish. Some of the

* Presented to the Society (Chicago) January 2, 1903. Received for publication May 30, 1903.

† *Linear Associative Algebra*, American Journal of Mathematics, vol. 4 (1884), p. 111.

classes on the main diagonal may contain no idempotent, in which case all the numbers of the class are nilpotent. The problem of determining the skew numbers is simple, and easily solved when the *direct* numbers of the classes (11), $(22)\cdots(kk)$ are known. Thus the basal problem of linear associative algebra is to determine all nilpotent algebras. With these at hand we may build up algebras of any type by combining the nilpotent algebras and adding the proper skew units. The present paper is devoted to the consideration of this basal problem.

In part II, it is shown that in a *canonical* form, the units of any nilpotent algebra are expressible in the forms,

$$i_1, i_2, i_3, \cdots, i_{s_0}, j, i_1 j, i_2 j, \cdots, i_{s_0-r} j, j^2, i_1 j^2, \cdots,$$

where $j^m = 0$. In this order, the product of any two units is expressible linearly in terms of units which *follow* both factors. Second, it is shown that a set of units homologous to the units $i_1, i_2, \cdots, i_{s_0}$ may be chosen, which form a sub-algebra, the products being isomorphic with the products of $i_1, i_2, \cdots i_{s_0}$, so far as concerns terms involving only $i_1, i_2, \cdots, i_{s_0}$. This sub-algebra is the *base*, the unit j being *adjoined* to this base. From any base an increasing system of nilpotent algebras may be determined, each algebra in turn yielding others. Finally, in part III, certain applications are given to exemplify the method.

PART II.

§ 1. It has been shown in a previous paper * that any linear associative algebra can be brought to a form in which we may express any number thus:

$$\phi = \Sigma a_{ijk} \lambda_{ijk},$$

with the following conditions:

(1) for any given value of k, $i = 1, \cdots, s_k$; $j = 1, \cdots, s_0$;

(2) $$s_0 \geqq s_1 \geqq s_2 \cdots \geqq s_P;$$

(3) the number k is called the *weight* of the term to which it is attached; also, for λ's that have the subscripts i and j equal, there is in each case a maximum weight, called the *multiplicity* of the λ's corresponding, and represented by μ_i for the terms λ_{iik}; thus the terms λ_{iik} are $\lambda_{ii0}, \lambda_{ii1}, \lambda_{ii2}, \cdots, \lambda_{ii\mu_i}$; we have then for λ_{ijk}, $k \geqq 0$, and $\mu_i > k > \mu_i - \mu_j - 1$;

(4) finally

$$\lambda_{ijk} \cdot \lambda_{i'j'k'} = c\delta_{ji'} \lambda_{ij'k+k'},$$

where $\delta_{ji'} = 0$ if $i' \neq j$, $\delta_{ji'} = 1$ if $i' = j$; and further if $\mu_i > k + k' > \mu_i - \mu_{j'} - 1$, $c = 1$, while if $k + k' \geqq \mu_i$, $c = 0$.

* *Theory of linear associative algebra*, Transactions of the American Mathematical Society, vol. 4 (1903), pp. 251–287. This paper is cited hereafter as *Theory*.

(5) The numbers ϕ satisfy an equation which consists of factors of determinant form, of orders $w_1, w_2, \cdots, w_s$. These numbers are the *widths* of the factors, and w_r represents the number of multiplicities of a certain value μ_r, which are equal. The factor corresponding is a determinant of order w_r, affected with the exponent μ_r. The equation is then

$$\Pi \begin{vmatrix} a_{i_r i_r 0} - \phi, & a_{i_r i_r+1 0}, & \cdots & a_{i_r i_r+w_r-1 0} \\ a_{i_r+1 i_r 0}, & a_{i_r+1 i_r+1 0} - \phi, & \cdots & a_{i_r+1 i_r+w_r-1 0} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i_r+w_r-1 i_r 0}, & a_{i_r+w_r-1 i_r 0}, & \cdots & a_{i_r+w_r-1 . i_r+w_r-1 0} - \phi \end{vmatrix}^{\mu_r} = 0$$

$$(i_r = w_1\mu_1 + \cdots w_{r-1}\mu_{r-1};\ r = 1, 2, \cdots e).$$

The *degree* of this equation is the *degree of the algebra.*

It is also evident that if n is the order of the algebra

$$w_1\mu_1 + w_2\mu_2 + \cdots w_e\mu_e = n \quad \text{or} \quad n+1,$$

according as we have or have not a *modulus.* When the factors are all linear the algebra is *non-quaternionic* in SCHEFFERS's notation, of type $(1, 1, \cdots, 1)$ in the nomenclature used here; when there is a factor of determinant order two, it is *quaternionic*, or of type $(2, \cdots)$, and so for higher types.* The equation of the algebra determines some of the *units* that define the algebra, but in general there will be units not determined by the equation. If we cut out of the algebra the units which are determined by the characteristic equation, those which are left must form a *nilpotent* algebra, and for every number of such algebra we have $\phi^m = 0$, $(m \leqq n+1)$.

It was further shown, l. c., p. 275 that if we operate by ϕ on a certain set of n units which define the domain of the algebra, indicated by

$$\phi_{11}, \phi_{21}, \cdots, \phi_{s_0 1}, \phi_{12}, \phi_{22}, \cdots, \phi_{s_1 2}, \cdots, \cdots, \phi_{1\mu_1} - \phi_{w_1\mu_1}, \quad (\mu_1 > \mu_2 \cdots > \mu_e),$$

then

$$\phi \cdot \phi_{ij} = a_{1i0}\phi_{1j} + a_{2i0}\phi_{2j} + \cdots + a_{1i1}\phi_{1j+1} + \cdots + a_{1i2}\phi_{1j+2} + \cdots.$$

In the nilpotent algebra, ϕ_{11} is an idempotent modulus which is used simply to enable us to express the units in a matricular form. We may choose the notation so that $a_{ij0} = 0$ if $j \geqq i$.

§ 2. Since only for $\phi = \phi_{11}$ does $\phi_{ij}\phi$ contain ϕ_{ij}, from the equation last given; since $\phi_{ij}\phi_{11}$ contains ϕ_{ij}, whatever i and j are, and since from the formation of

* *Cf. with reference to this paper and the preceding:* E. CARTAN: *Les Groupes Bilinéaires et les Systemes de Nombres Complexes*, Annales de la Faculté des Sciences de Toulouse, vol. 12 (1898), pp. 81–99.

the original matrical forms for the study of associative algebras,* there is a modulus, in the algebra of order $n+1$, therefore ϕ_{11} is this modulus, and therefore $\phi_{11} = \lambda_{110} + \lambda_{220} + \lambda_{330} + \cdots \lambda_{s_0 s_0 0}$. The algebra *with* ϕ_{11} has the equation

$$(\phi - x_{11}\phi_{11})^m = 0 \qquad (m \leqq n+1).$$

The nilpotent algebra formed by the omission of ϕ_{11} has the equation

$$\phi^m = 0 \qquad (m \leqq n+1).$$

§ 3. Now because ϕ_{11} is the modulus

$$\phi_{ij}\phi_{11} = a_{110}\phi_{11} + a_{210}\phi_{21} + \cdots + a_{111}\phi_{12} + \cdots + a_{112}\phi_{13} + \cdots = \phi_{ij}.$$

Therefore

$$a_{ij\,j-1} = 1, \quad \text{and} \quad a_{ij\,s} = 0 \quad \text{if} \quad s \neq j-1.$$

Hence

$$\phi_{ij} = \lambda_{ij\,j-1} + \phi'_{ij}$$

and ϕ'_{ij} has no constituent of the form $a_{g1h}\lambda_{g1h}$. Obviously no number ϕ of the algebra contains a multiple of ϕ_{11}.

§ 4. It follows at once from the form of ϕ_{ij}, that since $\lambda_{ij\,j-1}$ must occur in one number at least, and since

$$\lambda_{ij\,j-1}\lambda_{111} = \lambda_{ij},$$

therefore

$$\phi_{ij}\phi_{12} = \phi_{ij+1} + \text{other terms possibly.}$$

But these other terms could arise only from the presence in ϕ_{ij} of terms having the form λ_{r1t}, and there are none such, by the equation above. Hence

$$\phi_{ij}\phi_{12} = \phi_{ij+1}.$$

Therefore the n units of the algebra are expressed completely by the non-vanishing products in the list:

$$\phi_{21},\ \phi_{31},\ \cdots,\ \phi_{s_0 1}; \qquad \phi_{12},\ \phi_{21}\phi_{12},\ \cdots,\ \phi_{s_0 1}\phi_{12}; \qquad \phi_{12}^2,\ \phi_{21}\phi_{12}^2,\ \cdots.$$

§ 5. If we express $\phi_{21}, \phi_{31}, \cdots, \phi_{s_0 1}$, in terms of the λ's, there will be certain terms of weight zero, others of a greater weight. If all those whose weight exceeds zero be cut off from these expressions, it is obvious that the expressions left, which we may represent by $\phi_{210}, \phi_{310}, \cdots, \phi_{s_0 10}$, form a nilpotent algebra of order $s_0 - 1$. By choosing the coefficients properly, it is evident from the form of the expression of the general number of such algebra, ϕ_0, that this nilpotent algebra may be made to be *any* nilpotent algebra of order $s_0 - 1$. Hence, by adjoining to any nilpotent algebra of order $s_0 - 1$, an additional unit,

* *Theory*, p. 254, § 2.

ϕ_{12}, $\phi_{12}^{p+1} = 0$, and also annexing to the product of any two units, ϕ_{i10}, ϕ_{j10}, terms of the form $a\phi_{i1} \cdot \phi_{12}^{r}$ (which is equivalent to changing ϕ_{i10} into ϕ_{i1}, ϕ_{j10} into ϕ_{j1}), we arrive at a new nilpotent algebra of a higher order, the order depending on the number of non-vanishing products $\phi_{i1}\phi_{12}^{r}$. The original algebra we call the *base*. It is *primary* when it is not derivable from any lower base. The only primary algebras are those of the form

$$\phi_{12},\ \phi_{12}^{2},\ \cdots,\ \phi_{12}^{r_0-1},\ \phi_{12}^{r_0} = 0.$$

These algebras are all of those whose *order* equals the *degree* less one.

The adjunction is most simply shown by an example. Suppose the base is the algebra composed of i, j, where $i^2 = j$, $i^3 = 0$. This takes the form

$$i = \phi_{210} = \lambda_{210} + \lambda_{320}, \qquad j = \phi_{310} = \lambda_{310}.$$

The adjoined unit ϕ_{12} will then give us, to fix our ideas, for all nilpotent algebras of order four, on this base, the frames:*

$$\begin{aligned}(a)\ \phi = {} & a_{210}\lambda_{210} \\ & + a_{310}\lambda_{310} + a_{210}\lambda_{320} \\ & + a_{111}\lambda_{111} + a_{121}\lambda_{121} \\ & + a_{211}\lambda_{211} + a_{221}\lambda_{221},\end{aligned} \qquad \begin{aligned}(b)\ \phi = {} & a_{210}\lambda_{210} \\ & + a_{310}\lambda_{310} + a_{210}\lambda_{320} \\ & + a_{111}\lambda_{111} \\ & + a_{112}\lambda_{112} + a_{122}\lambda_{122}.\end{aligned}$$

In (a),

$$\begin{aligned} i &= \phi_{21} = \lambda_{210} + \lambda_{320} + a\lambda_{121} + b\lambda_{221}, \\ j &= \phi_{31} = \lambda_{310} + a'\lambda_{121} + b'\lambda_{221}, \\ k &= \phi_{12} = \lambda_{111} + c\lambda_{221}, \\ l &= \phi_{21} \cdot \phi_{12} = \lambda_{211}. \end{aligned}$$

We now have $i^2 = \lambda_{310} + a\lambda_{111} + a\lambda_{221} + b\lambda_{211} = j + ak + bl$, and $a' = 0$, $a(c-1) + b' = 0$.

In (b),

$$\begin{aligned} i &= \phi_{21} = \lambda_{210} + \lambda_{320} + a\lambda_{122}, \\ j &= \phi_{31} = \lambda_{310} + b\lambda_{122}, \\ k &= \phi_{12} = \lambda_{111} + c\lambda_{122}, \\ l &= \phi_{12}^{2} = \lambda_{112}. \end{aligned}$$

We have, therefore, $i^2 = \lambda_{310} + a\lambda_{112} = j + al$, $b = 0$, $ki = cl$.

It is to be observed that if

$$\phi_{i10} \cdot \phi_{j10} = a\phi_{k10} + a'\phi_{k'10} + \cdots,$$

* *Theory*, p. 275, § 1.

we may have

$$\phi_{i1}\,\phi_{j1} = a\phi_{k1} + a'\phi_{k'1} + \cdots + b\phi_{21}\,\phi_{12} + \cdots + c\phi_{31}\,\phi_{12} + \cdots.$$

§ 6. We can reduce the multiplication table of any given nilpotent algebra to this canonical form by choosing as the first unit ϵ_1, say, any one of the numbers, preferably that one which has the highest power that does not vanish. Let its successive powers be the next units, $\epsilon_1^2, \epsilon_1^3, \cdots$. We may now choose from the remaining independent numbers one such that its product into ϵ_{11} shall be free from powers of ϵ_1, say ϵ_2,

$$\epsilon_2\epsilon_1 = \epsilon_2',$$

and

$$\epsilon_2'\epsilon_1 = \epsilon_2'',$$

etc. A third may be so chosen, and so on, until the domain is exhausted. This will give the column of products of ϵ_1 by each unit, in a canonical form, and the number ϵ_1 may be taken as ϕ_{12}. The others are then

$$\phi_{21},\quad \phi_{21}\phi_{12},\quad \cdots,$$

$$\phi_{31},\quad \phi_{31}\phi_{12},\quad \cdots.$$

$$\cdots\cdots\cdots$$

Thus, let the algebra be PEIRCE's (u_5)

	i	j	k	l	m
i	j	k	0	0	l
j	k	0	0	0	0
k	0	0	0	0	0
l	k	0	0	0	k
m	$j+rl$	$(1+r)k$	0	rk	$j+k+(2r-1)l$

$$r^3 = -1, \qquad r^2 - r + 1 = 0.$$

Put $m' = m - (1+r)i$, $l' = rl - rj$; then the first column is

	i
i	j
j	k
k	0
l'	0
m'	l'

According to the general principles worked out in the preceding paper,* the algebra must be a form of

$$\phi = a_{210}\lambda_{210} + a_{111}\lambda_{111} + a_{211}\lambda_{211} + a_{221}\lambda_{221} + a_{112}\lambda_{112} + a_{122}\lambda_{122} + a_{113}\lambda_{113} + a_{123}\lambda_{123}.$$

In fact,

$$\begin{aligned}
\phi_{12} = \ i \ &= \lambda_{111} - r^2\lambda_{221} - r\lambda_{122}, \\
\phi_{12}^2 = \ j \ &= \lambda_{112} - (1+r)\lambda_{123}, \\
\phi_{12}^3 = \ k \ &= \lambda_{113}, \\
\phi_{21} = m' &= \lambda_{210} + (r^2-1)\lambda_{221} - (1-r)\lambda_{122} + \lambda_{123}, \\
\phi_{21}\phi_{12} = l' &= \lambda_{211} + r\lambda_{123}.
\end{aligned}$$

Whence

$$\begin{aligned}
m &= \lambda_{210} - r\lambda_{122} + \lambda_{123} + (1+r)\lambda_{111}, \\
l &= -r^2\lambda_{211} + \lambda_{112} - r\lambda_{123}.
\end{aligned}$$

In any case, it is obvious that we may take any number as adjunct. It and its powers being removed from the domain, the remaining part of the domain can be represented by $\phi_{21}, \phi_{21}\phi_{12}, \phi_{21}\phi_{12}^2, \cdots \phi_{31}, \phi_{31}\phi_{12}, \cdots$. We then select as *base*, $\phi_{210}, \phi_{310}, \cdots$ differing from $\phi_{21}, \phi_{31}, \cdots$ only in the fact that their products do not contain the terms $\phi_{21}\phi_{12}, \cdots, \phi_{31}\phi_{12}, \cdots$. This base must then be expressed in the form (which is always possible and feasible),

$$\begin{aligned}
&a_{210}\lambda_{210} \\
&+ a_{310}\lambda_{310} + a_{320}\lambda_{320} \\
&+ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\
&+ a_{s_0 10}\lambda_{s_0 10} + a_{s_0 20}\lambda_{s_0 20} + \cdots + a_{s_0 s_0-1\,0}\lambda_{s_0 s_0-1\,0}.
\end{aligned}$$

§ 7. Another theorem is necessary to complete this statement of the possible forms of nilpotent numbers.

It is obvious from the λ forms, (since no one of them can be of the form λ_{iir}) that, if we choose any $n-1$ independent numbers (defining the field), say

$$\phi_1, \phi_2, \cdots, \phi_{n-1},$$

the products

$$\phi_r\phi_s \qquad (r = 1 \cdots n-1;\ s = 1 \cdots n-1),$$

are expressible in terms of at most $n-2$ independent numbers, let us say in terms of $n-1-h_1$.

*Cf. especially pp. 275–276, § 1.

The products $\phi_r\phi_s$ must then form an algebra of $n-1-h_1$ units. Let that part of the domain of the original algebra which is excluded from this sub-algebra, be defined by the numbers

$$\epsilon_1', \epsilon_2', \cdots, \epsilon_h'.$$

Let the sub-algebra be denoted by

$$\{\phi''\}.$$

Then

$$\epsilon_r'\epsilon_n' = \{\phi''\},$$

i. e., the product of any two units ϵ_r' is in the sub-algebra which excludes the units ϵ_r'.

All triple products $\phi_r\phi_s\phi_t$ also form a sub-algebra of at most $n-1-h_1-1$ units, say $n-1-h_1-h_2$. Let the domain of $\{\phi''\}$ excluded from this sub-algebra $\{\phi'''\}$ be defined by

$$\epsilon_1'', \epsilon_2'', \cdots, \epsilon_{h_2}''.$$

Then we have the equations

$$\epsilon_r'\{\phi''\} = \{\phi'''\}, \qquad \{\phi''\}\epsilon_r' = \{\phi'''\},$$

$$\epsilon_r'\epsilon_s' = \{\epsilon''\phi'''\}, \qquad \epsilon_r'\epsilon_s'' = \{\phi'''\}, \qquad \epsilon_s''\epsilon_r' = \{\phi'''\}.$$

We may so continue separating the field into classes of units, which give the table below, wherein

$$(\epsilon') = (\epsilon_1', \epsilon_2', \cdots, \epsilon_{h_1}'),$$

$$(\epsilon'') = (\epsilon_1'', \epsilon_2'', \cdots, \epsilon_{h_2}''),$$

	(ϵ')	(ϵ'')	(ϵ''')	$\cdots$
(ϵ')	$\{\phi''\}$	$\{\phi'''\}$	$\{\phi^{\text{iv}}\}$	$\cdots$
(ϵ'')	$\{\phi'''\}$	$\{\phi^{\text{iv}}\}$	$\cdots$	
(ϵ''')	$\{\phi^{\text{iv}}\}$	$\cdots$		

As a corollary * we may say that any nilpotent algebra may be written so that its table is

	1	2	3	$\cdots$	$n-2$	$n-1$
1	$(2\cdots n-1)$	$(3\cdots n-1)$	$(4\cdots n-1)$	$\cdots$	$(n-1)$	0
2	$(3\cdots n-1)$	$(4\cdots n-1)$	$\cdots$	$\cdots$	0	0
3	$(4\cdots n-1)$	. . . . .				
$\vdots$						
$n-2$	$(n-1)$	0	. . . . .			
$n-1$	0	0	. . . . .			

* CARTAN, loc. cit., pp. 13–33.

A further corollary: If the order is n, then for any $n+1$ numbers of the algebra,

$$\phi_1\phi_2\phi_3\cdots\phi_{n+1}=0.$$

Consequently, in the ϕ notation,

	ϕ_{21}	ϕ_{31}	$\cdots$	ϕ_{m1}	ϕ_{12}	$\phi_{21}\phi_{12}$	$\cdots$
ϕ_{21}	$(\phi_{31}\cdots)$	$(\phi_{41}\cdots)$	$\cdots$	$(\phi_{12}\cdots)$	$(\phi_{21}\phi_{12}\cdots)$	$(\phi_{31}\phi_{12}\cdots)$	$\cdots$
ϕ_{31}	$(\phi_{41}\cdots)$	. . . . .					
$\vdots$							
ϕ_{m1}	$(\phi_{12}\cdots)$	. . . . .					
ϕ_{12}	$(\phi_{21}\phi_{12}\cdots)$	. . . . .					
$\phi_{21}\phi_{12}$	$(\phi_{31}\phi_{12}\cdots)$	. . . . .					
$\vdots$							

This is equivalent to the statement that for any term λ_{ijk} occurring in the representation of $\phi_{g_1}\cdot\phi_{12}^{t}$, we must have

and

(1) $k \geqq t$,

(2) $i > g$, if $k = t$,

(3) $i \leqq g$, only when $k > t$.

For instance, if the frame is

210

310 320

410 420 430

111 121

211 221

311 321

112 122 132 142

then the possible forms are

$$\begin{aligned}\phi_{21} = (210) &+ a_{320}(320) + a_{420}(420) + a_{430}(430)\\ &+ a_{121}(121) + a_{221}(221) + a_{321}(321)\\ &+ a_{122}(122) + a_{132}(132) + a_{142}(142)\end{aligned}$$

$$\phi_{31} = (310) + a'_{420}(420) + a''_{430}(430) + \cdots$$

$$\phi_{41} = (410) + a''_{121}(121) + \cdots$$

$$\phi_{12} = (111) + a'''_{221}(221) + \cdots$$

$$\phi_{21}\phi_{12} = (211) + a^{\text{iv}}_{321}(321) + a^{\text{iv}}_{122}(122) + a^{\text{iv}}_{132}(132) + a^{\text{iv}}_{142}(142),$$

$$\phi_{31}\phi_{12} = (311) + a^{\text{v}}_{122}(122) + a^{\text{v}}_{132}(132) + a^{\text{v}}_{142}(142),$$

$$\phi_{12}^2 = (112).$$

Since $\phi_{12}^2 = [(111) + a'''_{221}(221) + \cdots]^2 = (112) + a'''_{121}a'''_{221}(122)$ it follows that $a'''_{121} = 0$ or $a'''_{221} = 0$, etc.

We are enabled to reduce the possible forms quite rapidly.

§ 8. It shonld also be remarked that the base is itself subject to the same laws as any other nilpotent algebra, and may be expressed in the same general forms.

Part III. Applications.

§ 1. As the first application of the preceding theory, let us consider *all nilpotent algebras for which the degree = order = n*.

There must then be at least one expression ϕ for which

$$\phi,\ \phi^2,\ \phi^3,\ \cdots,\ \phi^{n-1},$$

are all independent and do not vanish. Let this expression be ϕ_{12}, then the other unit gives a *base* of the form

	ϕ_{21}
ϕ_{21}	0

Hence the form of all such algebras is given by

$$\phi_{21} = \lambda_{210} + a\lambda_{12_{n-1}}, \qquad \phi_{12} = \lambda_{111} + b\lambda_{12_{n-1}}.$$

We need to note that

$$\phi_{21}^2 = a\lambda_{11_{n-1}} = a\phi_{12}^{n-1}, \qquad \phi_{21}\phi_{12} = 0, \qquad \phi_{12}\phi_{21} = b\phi_{12}^{n-1}.$$

These results agree with Peirce.*

§ 2. We can extend the applications by considering next the class of *all nilpotent algebras for which degree = order − 1 = n − 1*.

The base may be of order one or order two.

*American Journal of Mathematics, vol. 4 (1884), p. 116.

If the base is of order two, there are two cases, since the two units may combine in two ways. These two algebras are of the forms† corresponding to the two equations

$$\phi^3 = 0, \qquad \text{and} \qquad \phi^2 = 0.$$

In the first case the algebra is

$$\phi_{12}, \phi_{12}^2.$$

In the second case it is ϕ_{21}, ϕ_{12}. The base has the indefinite form

$$\begin{matrix} 210 \\ 310\ 320 \end{matrix}$$

and

$$(1)\quad \phi_{12} = 210 + 320, \qquad \phi_{12}^2 = 310.$$

$$(2)\quad \phi_{12} = 310, \qquad \phi_{21} = 210.$$

The first case is represented by the forms

$$\phi_{21} = \lambda_{210} + \lambda_{320} + a\lambda_{12n-2} + b\lambda_{13n-2},$$

$$\phi_{21}^2 = \phi_{31} = \lambda_{310} + a\lambda_{11n-2} + b\lambda_{12n-2},$$

$$\phi_{12} = \lambda_{111} + c\lambda_{12n-2} + d\lambda_{13n-2},$$

$$\phi_{21}\phi_{12} = 0 = \phi_{31}\phi_{12}.$$

Since

$$\phi_{12}\phi_{21} = c\lambda_{11n-2} + d\lambda_{12n-2} = c\phi_{12}^{n-2} + c\lambda_{11n},$$

therefore

$$d = 0, \qquad \phi_{12}\phi_{31} = 0.$$

In a table, these results would appear as follows:

	i	i^2	$\dots$	i^{n-2}	j	j^2
i	i^2	i^3	$\dots$	0	ci^{n-2}	0
i^2	i^3	i^4	$\dots$	0	0	0
$\vdots$	$\vdots$			$\vdots$	$\vdots$	
i^{n-2}	0	0		0	0	0
j	0	0		0	j^2	0
j^2	0	0		0	0	0

In the second case,

$$\phi_{21} = \lambda_{210} + a\lambda_{12n-2} + b\lambda_{13n-2},$$

$$\phi_{31} = \lambda_{310} + c\lambda_{12n-2} + d\lambda_{13n-2},$$

$$\phi_{12} = \lambda_{111} + e\lambda_{12n-2} + f\lambda_{13n-2}.$$

†American Journal of Mathematics, vol. 4 (1884), p. 121.

Therefore,

$$\phi_{21}^2 = a\phi_{12}^{n-2}, \qquad \phi_{31}^2 = d\phi_{12}^{n-2}, \qquad \phi_{21}\phi_{12} = 0 = \phi_{31}\phi_{12},$$

$$\phi_{12}\phi_{21} = e\phi_{12}^{n-2}, \qquad \phi_{12}\phi_{31} = f\phi_{12}^{n-2}.$$

Tabulated, we have

	i	i^2	i^3	...	i^{n-2}	j	k
i	i^2	i^3	i^4	...	0	ei^{n-2}	fi^{n-2}
i^2	i^3	i^4	i^5	...	0	0	0
i^3	i^4	i^5	i^6	...	0	0	0
⋮	⋮						
i^{n-2}	0	0	0	...	0	0	0
j	0	0	0	...	0	0	0
k	0	0	0	...	0	0	0

If the base is of order one,

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{12n-3} + c\lambda_{12n-2},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + b'\lambda_{12n-3} + c'\lambda_{12n-2},$$

$$\phi_{21}\phi_{12} = \lambda_{211} + a'b\lambda_{12n-2}.$$

From these equations,

$$\phi_{21}^2 = a\lambda_{211} + b\lambda_{11n-3} + c\lambda_{11n-2} + ab\lambda_{12n-2}.$$

This must be equal to $a\phi_{21}\phi_{12} + b\phi_{12}^{n-3} + c\phi_{12}^{n-2}$.

We have to consider separately the cases $n = 4$ and $n = 5$. *Let us suppose* $n > 5$; then

$$a\lambda_{211} + b\lambda_{11n-3} + c\lambda_{11n-2} + ab\lambda_{12n-2} = a\lambda_{211} + b\lambda_{11n-3} + c\lambda_{11n-2} + aa'b\lambda_{12n-2}.$$

Hence $ab(a' - 1) = 0$. Again,

$$\phi_{12}\phi_{21} = a'\lambda_{211} + b'\lambda_{11n-3} + c'\lambda_{11n-2} + ab'\lambda_{12n-2}$$

$$= a'(\lambda_{211} + a'b\lambda_{12n-2}) + b'\phi_{12}^{n-3} + c'\phi_{12}^{n-2};$$

therefore

$$a'^2 b = ab'.$$

Combining this with the preceding, we have

$$a'b = 0, \qquad \text{if} \qquad a = 0;$$

$$ab' = 0, \qquad \text{if} \qquad b = 0;$$

$$b = ab', \qquad \text{if} \qquad a' = 1.$$

The sub-cases are,

(1) $a = 0 = b$, giving $\phi_{21} = \lambda_{210} + c\lambda_{12n-2}$.

(2) $a = 0 = a'$ giving $\phi_{12} = \lambda_{111} + b'\lambda_{12n-3} + c'\lambda_{12n-2}$, $\phi_{21} = \lambda_{210} + b\lambda_{12n-3} + c\lambda_{12n-2}$.

(3) $b = 0 = b'$, giving $\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{12n-2}$, $\phi_{21} = \lambda_{210} + a\lambda_{221} + c\lambda_{12n-2}$.

(4) $a' = 1$, $b = ab'$, giving

$$\phi_{12} = \lambda_{111} + \lambda_{221} + b'\lambda_{12n-3} + c'\lambda_{12n-2},$$

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + ab'\lambda_{12n-3} + c\lambda_{12n-2}.$$

When $n = 4$, we have

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{121} + c\lambda_{122},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{122},$$

$$\phi_{21}\phi_{12} = \lambda_{211} + a'b\lambda_{122},$$

$$\phi_{12}\phi_{21} = a'\lambda_{211} + c'\lambda_{112} + b\lambda_{122} = a'(\lambda_{211} + a'b\lambda_{122}) + c'\lambda_{112}.$$

Therefore,

$$b(a' + 1)(a' - 1) = 0,$$

$$\phi_{21}^2 = a\lambda_{211} + b\lambda_{111} + c\lambda_{112} + ab\lambda_{122}$$

$$= a(\lambda_{211} + a'b\lambda_{122}) + b(\lambda_{111} + a'\lambda_{221} + c'\lambda_{122}) + c\lambda_{112}.$$

Therefore,

$$aa'b + bc' = ab, \qquad ba' = 0, \qquad b = 0.$$

Hence

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + c\lambda_{122} = j,$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{122} = i,$$

$$\phi_{21}\phi_{12} = \lambda_{211},$$

$$\phi_{12}\phi_{21} = a'\lambda_{211} + c'\lambda_{112},$$

or

	i	i^2	j	ji
i	i^2	0	$a'ji + c'i^2$	0
i^2	0	0	0	0
j	ji	0	$aji + ci^2$	0
ji	0	0	0	0

This type embraces a number of forms.*

* See PEIRCE's l_4, n_4, o_4, p_4, q_4, r_4, *loc. cit.*

When $n = 5$, we have

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{122} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + b'\lambda'_{122} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{211} + a'b\lambda_{123};$$

$$\phi_{21}^2 = a\lambda_{211} + b\lambda_{112} + c\lambda_{113} + ab\lambda_{123},$$

$$= a(\lambda_{211} + a'b\lambda_{123}) + b(\lambda_{112} + b'\lambda_{121} + a'b'\lambda_{123}) + c\lambda_{113}.$$

Therefore,

$$aa'b + bb' + a'bb' = ab.$$

Again,

$$\phi_{12}\phi_{21} = a'\lambda_{211} + b'\lambda_{112} + c'\lambda_{113} + b\lambda_{123} + ab'\lambda_{123}$$

$$= a'(\lambda_{211} + a'b\lambda_{123}) + b'(\lambda_{112} + b'\lambda_{123} + a'b'\lambda_{123}) + c'\lambda_{113}.$$

Therefore,

$$a'^2b + b'^2 + a'b'^2 = b + ab'.$$

(1) If $b = 0$,

$$b'^2(a' + 1) = ab'.$$

(1_1) If also $b' = 0$, both equations are satisfied. [Case 5_1].

(1_2) If $b' \neq 0$, $b'(a' + 1) = a$. [Case 5_2].

(2) If $b \neq 0$, $aa' + b' + a'b' - a = 0$, or $a = b'\dfrac{a' + 1}{1 - a'}$ if $a' \neq 1$.

(2_1) $a' \neq 1$, $b'^2(a' + 1) = (1 - a')(a'^2 - 1)b + b'^2(a' + 1)$, hence $a' + 1 = 0$, or $a' = -1$, and $a = 0$. [Case 5_3].

(2_2) If $a' = 1$, $bb' = 0$, $b' = 0$. [Case 5_4].

We have then the following cases:

5_1.

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{211}.$$

5_2.

$$\phi_{21} = \lambda_{210} + b'(a' + 1)\lambda_{221} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} + a'\lambda_{221} + b'\lambda_{122} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{221}.$$

5_3.

$$\phi_{21} = \lambda_{210} + b\lambda_{122} + c\lambda_{123},$$

$$\phi_{12} = \lambda_{111} - \lambda_{221} + b'\lambda_{122} + c'\lambda_{123},$$

$$\phi_{21}\phi_{12} = \lambda_{211} - b\lambda_{123}.$$

5_4.

$$\phi_{21} = \lambda_{210} + a\lambda_{221} + b\lambda_{122} + c\lambda_{123},$$
$$\phi_{12} = \lambda_{111} + \lambda_{221} + c'\lambda_{123},$$
$$\phi_{21}\phi_{12} = \lambda_{211} + b\lambda_{123}.$$

Or,* if $\phi_{12} = i$, $\phi_{21} = j$,

5_1	i	i^2	i^3	j	ji
i	i^2	i^3	0	$a'ji + c'i^3$	0
i^2	i^3	0	0	0	0
i^3	0	0	0	0	0
j	ji	0	0	$aji + ci^3$	0
ji	0	0	0	0	0

5_2	i	i^2	i^3	j	ji
i	i^2	i^3	0	$a'ji + b'i^2 + c'i^3$	$b'i^3$
i^2	i^3	0	0	$(a'+1)b'i^3$	0
i^3	0	0	0	0	0
j	ji	0	0	$(a'+1)b'ji + ci^3$	0
ji	0	0	0	0	0

5_3	i	i^2	i^3	j	ji
i	i^2	i^3	0	$-ji + b'i^2 + c'i^3$	$b'i^3$
i^2	i^3	0	0	0	0
i^3	0	0	0	0	0
j	ji	0	0	$bi^2 + ci^3$	bi^3
ji	0	0	0	$-bi^3$	0

5_4	i	i^2	i^3	j	ji
i	i^2	i^3	0	$ji + c'i^3$	0
i^2	i^3	0	0	0	0
i^3	0	0	0	0	0
j	ji	0	0	$aji + bi^2 + ci^3$	bi^3
ji	0	0	0	bi^3	0

§ 3. The next case would obviously be that in which

$$\phi_{12}^{n-3} = 0.$$

The discussion, however, must be postponed to a later paper. It involves no special difficulties.

§ 4. The problem of finding all algebras which satisfy a given equation of degree m, i. e. such that

$$\phi^m = 0,$$

is an interesting one. The simplest case is that in which

$$\phi^2 = 0.$$

The adjoined unit must be simply λ_{111}. The base units ϕ_i will then give either $\phi_i\lambda_{111} = 0$, or $\phi_i\lambda_{111} \neq 0$.

* These four types represent PEIRCE's t_5 to av_5 inclusive.

In the first case

$$\phi_i = \lambda_{i10} + L_i + a_2^{(i)}\lambda_{121} + a_3^{(i)}\lambda_{131} + \cdots,$$
$$\phi_{i'} = \lambda_{i'10} + L_{i'} + a_2^{(i')}\lambda_{121} + a_b^{(i')}\lambda_{131} + \cdots,$$

where L_i and $L_{i'}$ are the groups of terms in each of the units ϕ_i, $\phi_{i'}$, of the form λ_{rs0} where $s > 1$.

Hence

$$\phi_i\phi_{i'} = L_i\lambda_{i'10} + L_iL_{i'} + a_i^{(i)}\lambda_{111},$$
$$\phi_{i'}\phi_i = L_{i'}\lambda_{i10} + L_{i'}L_i + a_i^{(i')}\lambda_{111}.$$

But for any associative numbers which all satisfy the law $\phi^2 = 0$, we must have

$$\phi_i\phi_{i'} + \phi_{i'}\phi_i = 0.$$

Therefore,

$$a_{i'}^{(i)} + a_i^{(i')} = 0, \qquad \text{or} \qquad a_{i'}^{(i)} = -a_i^{(i')}.$$

In words, ϕ_i cannot contain $a\lambda_{1i'1}$ unless $\phi_{i'}$ contain $-a\lambda_{1i1}$.

It is evident the adjunction does not change the form of the base units in any way, unless for every term of weight unity, $a\lambda_{1j1}$, added to ϕ_{i10}, we add $-a\lambda_{1j1}$ to ϕ_{j10}. Hence we have the process of building up all algebras of this class, those of order $n+1$ from those of order n.

For examples, we have algebras as follows:

Order 1,

$$i = \lambda_{111}.$$

From (1), order 2,

$$i = \lambda_{210}, j = \lambda_{111}.$$

From (2), order 3,

(a), $\quad i = \lambda_{210}, j = \lambda_{310}, k = \lambda_{111}.$

(b), $\quad i = \lambda_{210} + a\lambda_{131}, j = \lambda_{310} - a\lambda_{121}, k = \lambda_{111}.$

From (3a), order 4,

(a), $\quad i = \lambda_{210}, j = \lambda_{310}, k = \lambda_{410}, l = \lambda_{111}.$

(b), $\quad i = \lambda_{210} + a\lambda_{131}, j = \lambda_{310} - a\lambda_{121}, k = \lambda_{410}, l = \lambda_{111},$

(c), $\quad i = \lambda_{210} + a\lambda_{131} + b\lambda_{141}, j = \lambda_{310} - a\lambda_{121} + b'\lambda_{141},$
$k = \lambda_{410} - b\lambda_{121} - b'\lambda_{131}, l = \lambda_{111}.$

From (3b),

(b'), $i = \lambda_{210} + a\lambda_{430}, j = \lambda_{310} - a\lambda_{420}, k = \lambda_{410}, l = \lambda_{111},$

(d), $i = \lambda_{210} + a\lambda_{430} + b\lambda_{131}, j = \lambda_{310} - a\lambda_{420} - b\lambda_{121}, k = \lambda_{410}, l = \lambda_{111}.$

The forms (b) and (b') are essentially the same.

From (4a), order 5,

(a), $i = \lambda_{210}, j = \lambda_{310}, k = \lambda_{410}, l = \lambda_{510}, m = \lambda_{111}.$

(b), $i = \lambda_{210} + a\lambda_{151}, j = \lambda_{310}, k = \lambda_{410}, l = \lambda_{510} - a\lambda_{121}, m = \lambda_{111}.$

(c), $i = \lambda_{210} + a\lambda_{151}, j = \lambda_{310} + b\lambda_{141}, k = \lambda_{410} - b\lambda_{131}, l = \lambda_{510} - a\lambda_{121},$
$m = \lambda_{111}.$

(d), $i = \lambda_{210} + a\lambda_{151} + a'\lambda_{141}, j = \lambda_{310} + b\lambda_{141} + b'\lambda_{151},$
$k = \lambda_{410} - a\lambda_{121} - b\lambda_{131}, l = \lambda_{510} - a\lambda_{121} - b'\lambda_{131}, m = \lambda_{111}.$

(e), $i = \lambda_{210} + a\lambda_{151} + a'\lambda_{141} + a''\lambda_{131}, j = \lambda_{310} + b\lambda_{141} + b'\lambda_{151} - a''\lambda_{121}$
$k = \lambda_{410} - a\lambda_{121} - b\lambda_{131}, l = \lambda_{510} - a\lambda_{121} - b'\lambda_{131}, m = \lambda_{111}.$

From (4b),

(f), $i = \lambda_{210} + a\lambda_{430} + b\lambda_{151}, j = \lambda_{310} - a\lambda_{420} + c\lambda_{151}, k = \lambda_{410},$
$l = \lambda_{510} - b\lambda_{121} - c\lambda_{131}, m = \lambda_{111}.$

(g), $i = \lambda_{210} + a\lambda_{430} + b\lambda_{151} + c\lambda_{131}, j = \lambda_{310} - a\lambda_{420} + d\lambda_{151} - c\lambda_{121},$
$k = \lambda_{410}, l = \lambda_{510} - b\lambda_{121} - c\lambda_{131}, m = \lambda_{111}.$

From (4c),

(h), $i = \lambda_{210} + a\lambda_{530} + b\lambda_{540}, j = \lambda_{310} - a\lambda_{520} + b'\lambda_{540},$
$k = \lambda_{410} - b\lambda_{520} - b'\lambda_{530}, l = \lambda_{510}, m = \lambda_{111}.$

(i), $i = \lambda_{210} + a\lambda_{530} + b\lambda_{540} + c\lambda_{151}, j = \lambda_{310} - a\lambda_{520} + b'\lambda_{540} + d\lambda_{151},$
$k = \lambda_{410} - b\lambda_{520} - b'\lambda_{530} + e\lambda_{151}, l = \lambda_{510} - c\lambda_{121} - d\lambda_{131} - e\lambda_{141},$
$m = \lambda_{111}.$

(j), $i = \lambda_{210} + a\lambda_{530} + b\lambda_{540} + c\lambda_{151} + c'\lambda_{141} + c''\lambda_{131},$
$j = \lambda_{310} - a\lambda_{520} + b'\lambda_{540} + d\lambda_{151} + d'\lambda_{141} - c''\lambda_{121},$
$k = \lambda_{410} - b\lambda_{520} - b'\lambda_{530} + e\lambda_{151} - d'\lambda_{131} - c'\lambda_{121},$
$l = \lambda_{510} - c\lambda_{121} - d\lambda_{131} - e\lambda_{141},$
$m = \lambda_{111}.$

From $4d$,

$$(k), \quad i = \lambda_{210} + a\lambda_{430} + b\lambda_{530} + c\lambda_{131} + c'\lambda_{141} + c''\lambda_{151},$$
$$j = \lambda_{310} - a\lambda_{420} - b\lambda_{520} + d\lambda_{141} + d'\lambda_{151} - c\lambda_{121},$$
$$k = \lambda_{410} + e\lambda_{151} - c'\lambda_{121} - d\lambda_{131},$$
$$l = \lambda_{510} - c''\lambda_{121} - d'\lambda_{131} - e\lambda_{141},$$
$$m = \lambda_{111}.$$

In the other class, when $\phi_i\phi_{12} \neq 0$, then

$$\phi_i = \lambda_{i10} + L_i + a_{21}^{(i)}\lambda_{121} + \cdots + a_{22}^{(i)}\lambda_{221} + \cdots$$

$$\phi_i\phi_{12} = \lambda_{i11} + \lambda_{i10}L_i + \cdots = -\phi_{12}\phi_i = -a'_{ii}\lambda_{i11} + \cdots.$$

Therefore,

$$a'_{ii} = -1, \qquad a'_{ij} = 0, \qquad i \neq j.$$

Hence

$$\phi_{12} = \lambda_{111} - \delta_1\lambda_{221} - \delta_2\lambda_{331}\cdots, \qquad \delta_1, \delta_2, \cdots = 1 \text{ or } 0.$$

As before, ϕ_i contains $a\lambda_{1i'1}$ only if $\phi_{i'}$ contains $-a\lambda_{1i1}$.

For example,

$$i = \lambda_{210} + \lambda_{431}, \qquad j = \lambda_{310} - \lambda_{421}, \qquad k = \lambda_{410},$$

$$l = \lambda_{111} - \lambda_{221} - \lambda_{331} - \lambda_{441}, \qquad m = \lambda_{211}, \qquad n = \lambda_{311}, \qquad p = \lambda_{411}.$$

Further developments on this line must be deferred. The next problem would evidently be the determination of the forms for the class of algebras satisfying the equation $\phi^3 = 0$. The construction of algebras with idempotents from the nilpotent forms is also a large part of the further development.

ON SOLUTIONS OF DIFFERENTIAL EQUATIONS WHICH POSSESS AN OSCILLATOIN THEOREM*

BY

HELEN A. MERRILL

§1. Introduction.

The object of this paper is the study of differential equations possessing certain properties, later defined, which may be called *Sturmian properties*, since they characterize equations first discussed by Sturm in his celebrated *Mémoire sur les équations linéaires du second ordre*, in the first volume of Liouville's Journal. This memoir of Sturm's is chiefly devoted to the study of the equation

$$\frac{d}{dx}\left[K(x,\lambda)\frac{dy(x,\lambda)}{dx}\right]+G(x,\lambda)y(x,\lambda)=0,$$

with a few slightly more general forms. At the close of the article he refers briefly to the fact that these results were first obtained by the use of equations in finite differences of the second order, but none of this work was ever published,† owing probably to the fact that at that time it was impossible to treat the subject with sufficient rigor by this method. In the present state of analysis Sturm's original method may easily be placed on a satisfactory basis as regards rigor, and it is found to possess certain marked advantages, due to the ease with which recurrent relations satisfying the necessary conditions are set up, and to the possibility of generalizing much of the work. Sturm's famous theorem in regard to the location of the real roots of an equation is only a result of the application of this general method to polynomials.

In order to establish on a rigorous basis the method of finite differences, use will be made of a recent theorem‡ of Painlévé and Picard in regard to the Cauchy-Lipschitz proof of the existence of a solution of a differential equation.

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† A partial reconstruction of the original method of Sturm, with proofs of several of the Sturmian theorems, was published in the Annals of Mathematics, January, 1902, by M. B. Porter.

‡ This theorem has reference to the uniformity of the convergence of the Cauchy algorithm and to the interval of the convergence of the resulting solution. See p. 432.

After recalling some familiar definitions and theorems of which use will be made, a method will be developed by which recurrent relations of the required character are constructed. Some properties of the functions which satisfy these relations will then be established, and it will be shown that these same properties characterize the differential equations, to which under certain conditions the recurrent relations lead.

§ 2. Preliminary Definitions and Statements.

Sturmian Sequence. A sequence of functions, $y_0(\lambda)$, $y_1(\lambda) \cdots y_n(\lambda)$, may be called *Sturmian* when the number of real roots of $y_n(\lambda)$ lying between any two real values of the argument λ is a function of the number of variations in the signs of the sequence lost or gained as λ increases between these two values.

The functional relation here employed is that in general use, viz., the number of real roots between any two real values of λ is equal to the difference between the number of variations of the sequence for these two values of λ. In order that a sequence may be Sturmian in this special sense the following conditions are sufficient:

I. 1. $y_0(\lambda)$ does not vanish throughout the interval considered.
2. Two consecutive functions cannot vanish for the same value of λ.
3. When an intermediate function vanishes, the two adjacent functions have unlike signs.

II. $y_n(\lambda)$ and $y_{n-1}(\lambda)$ have unlike signs just before $y_n(\lambda)$ vanishes, and like signs just after. A sufficient condition to secure this property is the following:

$$\frac{d}{d\lambda}\left[\frac{y_n(\lambda)}{y_{n-1}(\lambda)}\right]_{y_n(\lambda)=0} > 0.$$

Remark. In case the functions have no derivatives, let it be possible to write $y_n(\lambda)$ in the vicinity of the point a, at which it vanishes, in the form

$$y_n(\lambda) = (\lambda - a)\, T(\lambda),$$

where $T(\lambda)$ is continuous and does not vanish for $\lambda = a$. We then have

$$\frac{y_n(\lambda)}{y_{n-1}(\lambda)} = (\lambda - a)\frac{T(\lambda)}{y_{n-1}(\lambda)},$$

an expression which must be finite for values of λ in the vicinity of a, and vanish for $\lambda = a$. Condition II is then satisfied, provided in the vicinity of a,

$$\frac{T(\lambda)}{y_{n-1}(\lambda)} > 0.$$

The above conditions I and II will be referred to as the *first* and *second Sturmian conditions* respectively.

CHARACTER OF THE FUNCTIONS EMPLOYED. In all this work it will be assumed that the functions under consideration, as well as the coefficients of the homogeneous recurrent relation which any three consecutive functions must satisfy, are real, single-valued, continuous, differentiable functions of a real parameter λ, which is supposed in every case to lie within a certain definite interval, $\lambda_i \leqq \lambda \leqq \lambda_k$. It is to be noted that none of the functions involved are necessarily analytic.

NOTATION. For convenience in writing, the argument of the functions will in general be omitted, but a consistent notation will be employed—the first letters of the alphabet denoting functions independent of λ, all other letters denoting functions of λ.

§ 3. DEVELOPMENT OF A METHOD OF FORMING RECURRENT RELATIONS SUCH THAT THE FUNCTIONS SATISFYING THEM FORM A STURMIAN SEQUENCE.

Our first effort is to satisfy the second Sturmian condition, that is, to impose conditions sufficient to secure the increase of y_n/y_{n-1} with λ either throughout the interval considered or for values of λ for which y_n vanishes. It will appear later that condition I can then be readily imposed.

A homogeneous recurrent relation of the second order, $F(y_{m+1}, y_m, y_{m-1}) = 0$, can be written in the form

$$z_{m+1} = f(z_m) \qquad (m = 0, 1, 2 \cdots n-1),$$

where $z_{m+1} = y_{m+1}/y_m$, $z_m = y_m/y_{m-1}$. If now conditions be imposed sufficient to secure monotone increase of z_{m+1} as z_m increases monotone, the second Sturmian condition is evidently met. Our mode of procedure, briefly stated, is to select a simple expression connecting z_m and z_{m+1}, by which under certain conditions the second Sturmian condition is secured, and to operate upon it by linear transformations such that this condition is at every step conserved, while the first Sturmian condition also is ultimately satisfied. The simplest possible relation connecting z_m and z_{m+1} is of the linear fractional type:

$$z_{m+1} = \frac{K_m z_m + L_m}{M_m z_m + N_m}, \tag{1}$$

which may be regarded as a transformation of the z_m plane on the z_{m+1} plane. Conditions are to be imposed upon K_m, L_m, M_m, N_m, sufficient to secure monotone increase of z_{m+1} when z_m increases monotone, both z_{m+1} and z_m having a range of values from $-\infty$ to $+\infty$. That is, as z_m describes the real axis in the positive sense, z_{m+1} is to describe the real axis in its plane, also in the positive sense. This transformation may be broken up into the following steps:

$$(a) \qquad z_m^{(1)} = M_m z_m + N_m .$$

$$(b) \qquad z_m^{(2)} = \frac{1}{z_m^{(1)}} .$$

$$(c) \qquad z_{m+1} = \frac{L_m M_m - K_m N_m}{M_m} z_m^{(2)} + \frac{K_m}{M_m} .$$

We shall first develop in detail the recurrent relation of which the differential equation treated by STURM may be regarded as the limiting form, and then indicate briefly how in the same way more general forms may be obtained.

DEVELOPMENT OF STURM'S FORM. The condition of monotone increase of z_{m+1} with z_m is evidently met by the translation

$$(2) \qquad z_{m+1} = z_m + L_m$$

where L_m increases monotone with λ ($L_m' \geqq 0$). This is a parabolic transformation with its double point at infinity. Each value of z_{m+1} is obtained by a translation of the corresponding value of z_m along the real axis in the positive direction. Let now both planes be subjected to the inversion and reflexion,

$$z_{m+1} = \frac{1}{\bar{z}_{m+1}}, \qquad z_m = \frac{1}{\bar{z}_m} .$$

The real axis in each plane is preserved, but in the opposite sense, the upper half of the plane being mapped upon the lower half. Hence in the transformed equation,

$$\bar{z}_{m+1} = \frac{\bar{z}_m}{1 + L_m \bar{z}_m},$$

$\bar{z}_{m+1}$ decreases as $\bar{z}_m$ decreases and L_m increases. Monotone increase of $\bar{z}_{m+1}$ with $\bar{z}_m$ is secured by making the substitution $L_m = 1/K_m$ and imposing the condition $K_m' \geqq 0$. The resulting linear relation is

$$(3) \qquad z_{m+1} = \frac{K_m z_m}{z_m + K_m} .$$

Let now the z_{m+1} and z_m planes be transformed once more, using a linear relation by which merely condition II is preserved. Then in relation (1) the following conditions are sufficient:

$$\begin{cases} K_m N_m - L_m M_m > 0, \\ L_m' N_m - L_m N_m' \leqq 0 . \end{cases}$$

These conditions are obviously fulfilled in the transformation

$$(4) \qquad \begin{cases} z_m = K_m w_m - K_m, \\ z_{m+1} = K_{m+1} w_{m+1} - K_{m+1}, \end{cases}$$

provided $K_i > 0$, $K_i' \geqq 0$, $(i = m, m+1)$. Applying this transformation to (3),

$$K_{m+1} w_{m+1} - K_{m+1} = K_m - K_m / w_m .$$

This relation has the property that w_{m+1} increases as it vanishes, provided $K_m - K_m/w_m$ increases. Let $w_{m+1} = y_{m+1}/y_m$, $w_m = y_m/y_{m-1}$, the last equation becomes

$$K_{m+1}(y_{m+1} - y_m) = K_m(y_m - y_{m-1}), \tag{5}$$

where, if $y_m = 0$, $K_{m+1} y_{m+1} = -K_m y_{m-1}$, that is, y_{m+1} and y_{m-1} have opposite signs.

If in place of (3) we write

$$z_{m+1} = \frac{K_m z_m}{z_m + K_m} + N_m \qquad (N_m' \geqq 0) \tag{6}$$

condition II is still met after the additional translation. Making the same transformation on z_m and z_{m+1} as before in obtaining (5), the resulting form is one which will be referred to as STURM's form:

$$K_{m+1}(y_{m+1} - y_m) = K_m(y_m - y_{m-1}) + N_m y_m, \tag{7}$$

or, using the ordinary difference notation,

$$\Delta(K_m \Delta y_{m-1}) = N_m y_m . \tag{7}'$$

If we impose in both (5) and (7), as always, the condition that y_0 does not vanish throughout the interval, $y_0, y_1 \cdots y_n$ form a Sturmian sequence. Here, as in all later similar cases, where the increase of z_{m+1} is conditioned upon the increase of z_m, the condition of monotone increase must be laid upon z_1. Thus in this case we must have

$$z_1' = \left[\frac{K_1(y_1 - y_0)}{y_0}\right]' \geqq 0 .$$

Further for each value of λ it is necessary that one or more of the functions upon which conditions of monotone increase are laid shall actually increase. If, for some fixed value of λ, $[K_1(y_1 - y_0)/y_0]' = 0$, we require that for at least one value of $m (m = 1, 2 \cdots n-1) N_m' > 0$, or that for two consecutive values of m, or for $m = n-1$, $K_m' > 0$.

DEVELOPMENT OF MORE GENERAL RECURRENT RELATIONS. Let the hyperbolic transformation used to secure the first Sturmian condition in (6) be

$$z_m = L_m w_m - K_m \qquad (L_m > 0), \tag{8}$$

of which (4) is a special form when $L_m = K_m$. The resulting recurrent relation is

$$L_{m+1} y_{m+1} - K_{m+1} y_m = \frac{K_m}{L_m}(L_m y_m - K_m y_{m-1}) + N_m y_m , \tag{9}$$

$$\Delta\{L_m \Delta y_{m-1} + (L_m - K_m) y_{m-1}\}$$

$$(9)' \qquad = \frac{K_m - L_m}{L_m}\{L_m \Delta y_{m-1} + (L_m - K_m) y_{m-1}\} + N_m y_m.$$

Here again both Sturmian conditions are satisfied, provided

$$L_m > 0, \quad K_m' \geqq 0, \quad N_m' \geqq 0, \quad \left[\frac{L_1 y_1 - K_1 y_0}{y_0}\right]' \geqq 0.$$

Instead of beginning with the parabolic substitution (2), we may use the general real elliptic form with fixed points $A_m^{(1)} = B_m + iC_m$, $A_m^{(2)} = B_m - iC_m$:

$$(10) \qquad z_{m+1} = \frac{(K_m + 2B_m) z_m - (B_m^2 + C_m^2)}{z_m + K_m}.$$

Forming the derivative with regard to λ, we have

$$z_{m+1}' = \frac{K_m'\{(z_m - B_m)^2 + C_m^2\} + z_m'\{(K_m + B_m)^2 + C_m^2\}}{(z_m + K_m)^2}.$$

Here z_{m+1} increases with λ, provided K_m and z_m increase. The same transformations already employed upon (5) give the following form:

$$(11) \quad K_{m+1}(y_{m+1} - y_m) = (K_m + 2B_m)(y_m - y_{m-1}) - \frac{B_m^2 + C_m^2}{K_m} y_{m-1} + N_m y_m.$$

Both Sturmian conditions are satisfied, provided the same conditions are imposed as in (7). It is to be noted that (7) is a special form of both (9) and (11). By writing $C_m = 0$ in (10) we have the most general parabolic substitution with double point at B_m. Still another form may be obtained by transforming (10) by means of (8).

The hyperbolic transformation is found to be useful only as an intermediate process in securing the first Sturmian property, under the forms already employed in transforming (6).

In the general linear relation (1) we find that monotone increase of z_{m+1} with z_m throughout the interval is secured only by imposing such complicated conditions as make this form too cumbrous for use.

Other recurrent relations may, however, be obtained by combining forms known to satisfy condition II. Consider for example the recurrent relation

$$(12) \qquad z_{m+1} = \frac{(K_m + 2B_m) z_m - (B_m^2 + C_m^2)}{z_m + K_m} + A_m z_m + N_m,$$

where the following conditions are fulfilled:

$$K_m > 0, \qquad K_m' \geqq 0, \qquad A_m \geqq 0, \qquad N_m' \geqq 0.$$

It is evident that z_{m+1} increases monotone with z_m. Employ now transformation (4), which gives the relation

$$\frac{K_{m+1}(y_{m+1}-y_m)}{y_m} = \frac{K_m(K_m+2B_m)(y_m-y_{m-1})-(B_m^2+C_m^2)y_{m-1}}{K_m y_m} + \frac{A_m K_m(y_m-y_{m-1})}{y_{m-1}} + N_m. \tag{13}$$

If $y_m = 0$, we have

$$K_m K_{m+1} y_{m+1} = -\{(K_m+B_m)^2 + C_m^2\} y_{m-1}$$

and both Sturmian conditions are satisfied.

Other such combinations may readily be made, leading to more or less complicated relations of which (7) will always be found to be a special form. If (12) were transformed by means of (8), the resulting relation would contain both (9) and (13) as special cases.

§ 4. Properties of a Sturmian Sequence.

Having thus indicated the method by which the desired recurrent relations are set up, we proceed to a consideration of some of the more important properties of the functions which satisfy these relations.

Graphical Illustration. The variations in the signs of the sequence $y_0, y_1 \cdots y_n$ may be illustrated for any fixed value of λ, as λ_p, by plotting the points whose rectangular coördinates are $x = m$, $y = y_m(\lambda_p)$, $(m = 0, 1 \cdots n)$. If the consecutive points whose abscissas are $0, 1 \cdots n$ be joined, the broken line thus formed cuts the X axis in points which in number and order correspond to the variations in the signs of the functions. These points of intersection will be referred to as the v-points of the sequence.

Linear Dependence. Two solutions of a recurrent relation, $y_m^{(1)}$ and $y_m^{(2)}$, are said to be linearly dependent when two constants c_1 and c_2 exist, such that $c_1 y_m^{(1)} + c_2 y_m^{(2)} = 0$, $m = 0, 1 \cdots n$.

The two following theorems will be needed for later proofs.

Theorem A. *Let $y_m^{(1)}$ and $y_m^{(2)}$ denote two solutions of a linear difference equation of the second order. If the coefficient of the y of highest index does not vanish, the necessary and sufficient condition for the linear dependence of the two solutions is $y_1^{(1)} y_0^{(2)} - y_1^{(2)} y_0^{(1)} = 0$.*

Theorem B. *Any solution can be expressed as a linear function of two linearly independent solutions.*

The following theorems are stated for the sake of definiteness for a set of functions satisfying the recurrent relation (9), but with certain obvious modifications, often simplifications, they hold for any similar relations.

$$L_{m+1} y_{m+1} - K_{m+1} y_m = \frac{K_m}{L_m}(L_m y_m - K_m y_{m-1}) + N_m y_m. \tag{9}$$

Let the following conditions be satisfied, the equality signs not holding simultaneously for all values of m. See p. 427.

$$L_m > 0,\ K_m' \geqq 0,\ N_m' \geqq 0,\ \left[\frac{L_1 y_1 - K_1 y_0}{y_0}\right]' \geqq 0.$$

Then, according to the previous work, $(L_n y_n - K_n y_{n-1})/y_{n-1}$ increases continually with λ between two roots of y_{n-1}, y_n/y_{n-1} increases when y_n vanishes, and $y_0, y_1 \cdots y_n$ form a Sturmian sequence.

THEOREM I. *A variation is lost in the sequence $y_0, y_1 \cdots y_n$ whenever y_n vanishes.*

THEOREM II. *The roots of y_n and of y_{n-1} separate each other; also the roots of $L_n y_n - K_n y_{n-1}$ and of y_{n-1}; also the roots of $L_n y_n - K_n y_{n-1} + M_n y_{n-1}$ and of y_{n-1}, provided $M_n' \geqq 0$.*

For $(L_n y_n - K_n y_{n-1} + M_n y_{n-1})/y_{n-1}$ increases continually with λ as λ passes from one root of y_{n-1} to the next, and assumes in this interval all real values from $-\infty$ to $+\infty$. The numerator therefore vanishes once in every such interval, and, as it can have no multiple roots, only once. If $M_n = K_n$, the first part of the theorem is proved; if $M_n = 0$, the second part is proved.

THEOREM III. *If $y_m^{(1)}$ and $y_m^{(2)}$ are any two linearly independent solutions of* (9) *satisfying the condition*

$$\left[\frac{L_1 y_1^{(i)} - K_1 y_0^{(i)}}{y_0^{(i)}}\right]' \geqq 0 \qquad (i = 1, 2).$$

their roots separate each other, provided K_m does not vanish throughout the interval considered.

This is proved by showing that $y_n^{(1)} y_{n-1}^{(2)} - y_n^{(2)} y_{n-1}^{(1)}$ cannot vanish or change sign. On substituting two consecutive roots of one solution this fact is contradicted, unless the other solution changes sign between these two roots.

THEOREM IV. *If $L_m' \geqq 0$, the v-points of the sequence remain stationary or move to the right as λ increases.*

Let a v-point lie between y_i and y_{i-1}. From the fact of the monotone increase of $L_m y_m - K_m y_{m-1}/y_{m-1}$ we have

$$\left[\frac{L_i y_i - K_i y_{i-1}}{y_{i-1}}\right]' \geqq 0.$$

Hence

$$L_i \left(\frac{y_i}{y_{i-1}}\right)' + L_i' \frac{y_i}{y_{i-1}} \geqq K_i' \geqq 0.$$

It follows that $(y_i/y_{i-1})' \geqq 0$, and as λ increases the v-point remains stationary or shifts toward the right.

THEOREM V. *Let $y_m^{(1)}$ and $y_m^{(2)}$ be any two linearly independent solutions of* (9), *and let the only conditions imposed be*

$$y_0^{(1)} \neq 0, \qquad y_0^{(2)} \neq 0, \qquad L_m L_{m-1} > 0, \qquad K_m \neq 0.$$

For any given value of λ *the v-points of the sequences* $y_0^{(1)}, y_1^{(1)} \cdots y_n^{(1)}$ *and* $y_0^{(2)}, y_1^{(2)} \cdots y_n^{(2)}$ *separate each other.*

Consider the solution $y_m = ty_m^{(1)} + y_m^{(2)}$. By a suitable variation of the parameter t the v-points of the sequence $y_0, y_1 \cdots y_n$ may be made to move, always in the same sense, from coincidence with the v-points of the sequence $y_0^{(1)}, y_1^{(1)} \cdots y_n^{(1)}$ into coincidence with those of the sequence $y_0^{(2)}, y_1^{(2)} \cdots y_n^{(2)}$, without once crossing a v-point of either sequence. Hence the v-points of the two given sequences separate each other.

THEOREM VI. *Let* λ_1 *and* λ_2 *be any two values of* $\lambda (\lambda_1 < \lambda_2)$, *and let* $(K_m^2/L_m)' \geqq 0$. *Between two consecutive v-points of the sequence* $y_0(\lambda_1), y_1(\lambda_1) \cdots y_n(\lambda_1)$ *cannot lie more than one v-point of the sequence* $y_0(\lambda_2), y_1(\lambda_2) \cdots y_n(\lambda_2)$, *and between two consecutive v-points of the latter sequence lies always at least one v-point of the former.*

To establish this theorem it may be proved that if a solution has for a given value of λ, as λ_1, the two consecutive v-points a and b, b lying to the right of a, a solution may be found which for every value of λ has the v-point a, and that its next v-point to the right must lie to the right of b. For the value $\lambda = \lambda_2$ the v-points of this solution and of the given solution must separate each other by Theorem V. The theorem then follows immediately.

THEOREM VII. THEOREM OF OSCILLATION. *If in the interval* $(\lambda_i \lambda_k)$ L_m *and* K_m *never exceed in absolute value certain fixed quantities, large at pleasure, if also* $N_m(\lambda_i) \leqq -P^2$, $N_m(\lambda_k) \geqq Q^2$, *and* P^2 *and* Q^2 *are sufficiently large, there is one and only one value of* λ $(\lambda_i \leqq \lambda \leqq \lambda_k)$ *such that a solution* y_m *exists, satisfying the following conditions:*

1. *The number of variations in the sequence* $y_1, y_2 \cdots y_n$ *is equal to an assigned integer* r $(r < n)$;

2. $$\frac{L_1 y_1 - K_1 y_0}{y_0} = a;$$

3. $$\frac{L_n y_n - K_n y_{n-1}}{y_{n-1}} = b;$$

a and b being any real constants.

Using condition 2, y_1, and therefore all y's of higher index, can be determined except for the non-vanishing factor y_0. Owing to the conditions imposed upon the coefficients, there are $n-1$ variations in the sequence $y_1, y_2 \cdots y_n$ when $\lambda = \lambda_i$ and none when $\lambda = \lambda_k$. Since $(L_n y_n - K_n y_{n-1})/y_{n-1}$ increases continually with λ between two consecutive roots of y_{n-1}, condition 3 will be satisfied once and only once in every such interval.

§ 5. The Differential Equation.

A recurrent relation of the second order may be written as an equation in finite differences of the second order, and this, under certain conditions, has as its limiting form a differential equation of the second order. The Cauchy-Lipschitz proof of the existence of a solution of a differential equation within a certain interval treats the differential equation as the limit of a difference equation. Painlévé and Picard have shown * that the solution converges uniformly to its limit and that the limit represents the solution as long as it remains continuous. It follows that theorems proved for $y_n(\lambda)$ in the difference equation can be extended without further proof to $y(x, \lambda)$ in the differential equation which is its limiting form. In particular, theorems proved in § 4 go over at once into their well-known analogues for the differential equation.

Consider for example equation (13), which may be written in the form

$$(13)' \quad y_{m-1}\Delta\{K_m\Delta y_{m-1}\} - \left\{ y_{m-1}\left(2B_m + \frac{B_m^2 + C_m^2}{K_m} + A_m K_m\right) + A_m K_m \Delta y_{m-1}\right\}\Delta y_{m-1} = y_m y_{m-1}\left\{N_m - \frac{B_m^2 + C_m^2}{K_m}\right\}.$$

On making the usual hypotheses in regard to the limits of the functions involved, in order to pass to the differential equation, we find that A_m, B_m and C_m must approach zero, and that different results may be obtained, depending upon the manner of approach. If it is sufficiently rapid, Sturm's equation results. If on the other hand A_m, B_m and C_m contain δ in as low a power as possible (δ being one of the equal subdivisions of the x interval, $x = x_0 + m\delta$), other forms are obtained. Thus if we suppose the limits to be the following:

$$\lim_{\delta=0}\frac{A_m}{\delta} = A(x), \qquad \lim_{\delta=0}\frac{2B_m}{\delta} = B(x), \qquad \lim_{\delta=0}\frac{B_m^2 + C_m^2}{\delta} = C(x),$$

from (13)′ we have the differential equation

$$(14) \qquad \frac{d}{dx}\left(K\frac{dy}{dx}\right) - \left(B + \frac{C}{K} + AK\right)\frac{dy}{dx} = Ny.$$

This possesses the same general properties as Sturm's equation, providing the following conditions are met:

$$K > 0,\ K' \geqq 0,\ A \geqq 0,\ C \geqq 0,\ N' \geqq 0,\ \left[\begin{matrix} K\dfrac{dy}{dx} \\ y \end{matrix}\right]'_{x=x_0} \geqq 0,\dagger$$

not all the signs of equality holding simultaneously.

* Bulletin de la Société Mathématique de France, vol. 27 (1899), p. 150. Comptes Rendus, vol. 128 (1899), p. 1363.

† The sign of differentiation in each case refers as before to λ.

From (14) six special forms may be derived by letting one or two of the coefficients A, B, C, vanish.

Consider further equation (9)′, which may be simplified by writing $L_m - K_m = M_m$, whence $L'_m \geqq M'_m$. This leads directly to the differential equation

$$\frac{d}{dx}\left(L\frac{dy}{dx} + My\right) = \frac{M}{L}\left(L\frac{dy}{dx} + My\right) + Ny, \tag{15}$$

with the conditions

$$L > 0,\ L' \geqq M',\ N' \geqq 0,\ \left[\frac{L\dfrac{dy}{dx} + My}{y}\right]'_{x=x_0} \geqq 0,$$

not all the signs of equality holding simultaneously.

The theorems proved above for equation (9) are evidently immediately applicable to equation (15).

It may readily be shown that each of the equations (14) and (15), together with the six special forms referred to, is more general than the equation discussed by STURM. For, by the transformation $y = uv$ and by removal of the term in dv/dx, each may be reduced to the general type to which STURM's equation belongs:

$$\frac{d}{dx}\left(R\frac{dv}{dx}\right) = Sv.$$

But in every case it is apparent that the conditions laid upon the function S do not ensure its monotone increase with λ, while in the case of (15) the same statement holds true for R. It follows that not simply equations (14) and (15), but many specialized forms of the former as well, are more general than the equation of STURM, which appears as a special case under each.

Hence the properties proved in STURM's memoir for a special equation are seen to belong to a more extensive class of equations, which are derived by most simple methods, and which clearly all possess an oscillation theorem in the Sturmian sense.

YALE UNIVERSITY,
May, 1903.

ON THE REDUCIBILITY OF LINEAR GROUPS*

BY

LEONARD EUGENE DICKSON

The object of this note is a two-fold generalization of LOEWY's theorem proved in these Transactions, vol. 4, pp. 171–177. His theorem may be conveniently stated as follows: If R is the domain of all real numbers and C the domain of all complex numbers, any group of linear homogeneous transformations with coefficients in R which is irreducible in R, but reducible in C, can be transformed linearly into a decomposable group $\left(\begin{smallmatrix} G & 0 \\ 0 & \bar{G} \end{smallmatrix}\right)$, where G and $\bar{G}$ are two groups irreducible in C, with coefficients not all in R, such that the coefficients in every transformation of $\bar{G}$ are the conjugate imaginaries of the corresponding coefficients for G.

In seeking a generalization, we note that the domain C may be considered as derived from R by the adjunction of a root i of the quadratic equation $x^2+1=0$ belonging to and irreducible in R. For the generalization, R is replaced by a general domain F (or field not having a modulus) and $R(i)$ is replaced by the domain $F(\rho_0)$ given by the extension of F by the adjunction of a root ρ_0 of an equation $f(x)=0$ of degree r belonging to and irreducible in F. The generalization will therefore be two-fold. Let the roots of $f(x)=0$ be $\rho_0, \rho_1, \cdots, \rho_{r-1}$. If G_{11} is a group of transformations with coefficients $C_{ij}(\rho_0)$ in the domain $F(\rho_0)$, let $G_{11}^{(s)}$ denote the group of transformations with the coefficients $C_{ij}(\rho_s)$; in particular, $G_{11}^{(0)}=G_{11}$. The coefficients of $G_{11}, G_{11}', \cdots, G_{11}^{(r-1)}$ are thus conjugate with respect to F. The generalized theorem is as follows:

Let G be a group of linear homogeneous transformations with coefficients in a domain F, such that G is irreducible in F but is reducible in the domain $F(\rho_0)$ given by the extension of F by the adjunction of a root ρ_0 of an equation belonging to and irreducible in F and having as its roots $\rho_0, \rho_1, \cdots, \rho_{r-1}$. Then G can be transformed linearly into a decomposable group †

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† When the irreducible equation is a normal equation, the groups $G_{11}^{(s)}$ $(s=0, 1, \cdots, r-1)$ are all irreducible in the same (normal) domain. LOEWY's case furnishes an example.

$$\begin{matrix} G_{11} & 0 & 0 & \cdots & 0 \\ 0 & G'_{11} & 0 & \cdots & 0 \\ 0 & 0 & G''_{11} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & G^{(r-1)}_{11} \end{matrix}$$

where $G^{(s)}_{11}$ is a group irreducible in $F(\rho_s)$ with coefficients not all in F, and $G_{11}, G'_{11}, \cdots, G^{(r-1)}_{11}$ are conjugate with respect to F.

The proof starts as in LOEWY, §1. The first variation * occurs at the bottom of p. 173; we now take r-fold decomposable matrices

$$H = \begin{bmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & G \end{bmatrix}, \qquad Q = \begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & P' & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & P^{(r-1)} \end{bmatrix}.$$

Corresponding changes are to be made in the first two statements on p. 174. Thus, the diagonal groups in (6) are to be replaced by

$$G_{11}, G_{22}, G'_{11}, G'_{22}, G''_{11}, G''_{22}, \cdots, G^{(r-1)}_{11}, G^{(r-1)}_{22}.$$

In place of the transformation † (7), we have

$$(7') \qquad y_{jk} = \sum_{i=1}^{n} C^{(j)}_{ki} y^*_{ji} \qquad (k=1, \cdots, n; j=0, \cdots, r-1),$$

where $C^{(j)}_{ki}$ is a rational function of ρ_j with coefficients in F, and

$$(7_a) \qquad C^{(j)}_{ki} = 0 \qquad (k=1, \cdots, m; i=m+1, \cdots, n; j=0, \cdots, r-1).$$

Introduce two pairs each of rn new variables defined by

$$(8') \qquad y_{sk} = Y_{0k} + \rho_s Y_{1k} + \rho_s^2 Y_{2k} + \cdots + \rho_s^{r-1} Y_{r-1\,k} \qquad \begin{pmatrix} s=0, \cdots, r-1 \\ k=1, \cdots, n \end{pmatrix}.$$

$$(8'_1) \qquad y^*_{sk} = Y^*_{0k} + \rho_s Y^*_{1k} + \rho_s^2 Y^*_{2k} + \cdots + \rho_s^{r-1} Y^*_{r-1\,k}$$

This may be done since the determinant

$$\Delta \equiv \begin{vmatrix} 1 & \rho_0 & \rho_0^2 & \cdots & \rho_0^{n-1} \\ 1 & \rho_1 & \rho_1^2 & \cdots & \rho_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \rho_{r-1} & \rho_{r-1}^2 & \cdots & \rho_{r-1}^{n-1} \end{vmatrix} = \Pi(\rho_i - \rho_j) \neq 0.$$

* The statement on p. 173, lines 7–8, is apparently not used later; a proof follows readily from the main theorem under consideration.

† LOEWY's notation is unwieldy even in his simple case. I write y_{0k}, y_{1k} for his y_k, z_k. The transformed variables are marked * instead of being primed.

Solving (8′) for fixed k, while $s = 0, \cdots, r-1$, we get

$$(e) \qquad \Delta\, Y_{tk} = \sum_{s=0}^{r-1} (-1)^s D_{ts} y_{sk} \qquad (t = 0, 1, \cdots, r-1),$$

where

$$D_{ts} \equiv \begin{vmatrix} 1 & \rho_0 & \cdots & \rho_0^{t-1} & \rho_0^{t+1} & \cdots & \rho_0^{r-1} \\ 1 & \rho_1 & \cdots & \rho_1^{t-1} & \rho_1^{t+1} & \cdots & \rho_1^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \rho_{s-1} & \cdots & \rho_{s-1}^{t-1} & \rho_{s-1}^{t+1} & \cdots & \rho_{s-1}^{r-1} \\ 1 & \rho_{s+1} & \cdots & \rho_{s+1}^{t-1} & \rho_{s+1}^{t+1} & \cdots & \rho_{s+1}^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \rho_{r-1} & \cdots & \rho_{r-1}^{t-1} & \rho_{r-1}^{t+1} & \cdots & \rho_{r-1}^{r-1} \end{vmatrix}.$$

Substituting for y_{sk} in (e) its value from (7′) and then eliminating y_{si}^* by $(8_1')$, we obtain

$$(9) \qquad Y_{tk} = \sum_{\substack{i=1,\cdots,n \\ l=0,\cdots,r-1}} \alpha_{li}^{tk} Y_{li}^* \qquad (k = 1, \cdots n;\ t = 0, \cdots r-1),$$

where

$$\alpha_{li}^{tk} \equiv \frac{(-1)^t}{\Delta} \sum_{s=0}^{r-1} (-1)^s D_{ts} C_{ki}^{(s)} \rho_s^l.$$

The coefficients of transformation (9) *belong to the domain* F. It suffices to show that each α_{li}^{tk} is unaltered by the interchange of ρ_0 with ρ_j (j being any one of the series $1, 2, \cdots, r-1$), since it is then a symmetric function of $\rho_0, \rho_1, \cdots, \rho_{r-1}$ with coefficients in F. To show that, for example, it is unaltered by the interchange of ρ_0 with ρ_1, we note that under this interchange, D_{t0} and D_{t1} are interchanged, $D_{ts}(s > 1)$ is changed into $-D_{ts}$ while $C_{ki}^{(0)} \equiv C_{ki}(\rho_0)$ and $C_{ki}^{(1)} \equiv C_{ki}(\rho_1)$ are interchanged, and $C_{ki}^{(s)}(s > 1)$ is unaltered. Hence the factor of α given by the sum is changed in sign; likewise the factor $1/\Delta$.

Moreover, from (7_a) follows at once

$$\alpha_{li}^{tk} = 0 \qquad (i = m+1, \cdots, n;\ k = 1, \cdots, m;\ t, l = 0, 1, \cdots, r-1).$$

The group of transformations (9) is therefore of LOEWY's form (10), $\bar{H}_{11}$ being always a matrix of rm rows and rm columns. The proof is then readily completed as in LOEWY's case (bottom of p. 175 and 176).

THE UNIVERSITY OF CHICAGO,
April 27, 1903.

SEMIREDUCIBLE HYPERCOMPLEX NUMBER SYSTEMS*

BY

SAUL EPSTEEN

INTRODUCTION.

It is customary to classify hypercomplex number systems as *reducible* and *irreducible* and, owing to the theorem that a reducible system can always be built up out of irreducible ones, the latter only are enumerated. In the following paper, in which only systems with modulus are considered, the irreducible systems are further classified by means of their groups as "semireducible of the first kind" (§1), "semireducible of the second kind" (§2) and "absolutely irreducible" (§3).

§1.

If

$$e_1 \cdots e_m e_{m+1} \cdots e_n \tag{1}$$

are the n units of a hypercomplex number system containing a modulus, then

$$x = \sum_{i_1=1}^{n} x_{i_1} e_{i_1}, \qquad y = \sum_{i_2=1}^{n} y_{i_2} e_{i_2}$$

are the general numbers of the system. † The product must also be a number of the system

$$x' = xy = \sum_{i_3=1}^{n} x'_{i_3} e_{i_3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} x_{i_1} y_{i_2} e_{i_1} e_{i_2};$$

and, since

$$e_{i_1} e_{i_2} = \sum_{i_3=1}^{n} \gamma_{i_1 i_2 i_3} e_{i_3},$$

$$\sum_{i_3=1}^{n} x'_{i_3} e_{i_3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \gamma_{i_1 i_2 i_3} y_{i_2} x_{i_1} e_{i_3}.$$

Hence we obtain in the form

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† The scheme of subscripts employed in this paper is the following: The i's, viz., $i_1, i_2, i_3 = 1, \cdots, n$; the j's; $j_1, j_2, j_3 = 1, \cdots, m$; the k's, $k_1, k_2, k_3 = m+1, \cdots, n$.

$$G \equiv x'_{i_3} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \gamma_{i_1 i_2 i_3} y_{i_2} x_{i_1} \qquad (i_3 = 1, \cdots, n), \tag{2}$$

the group of the system (1),* the x's being regarded as variables, the y's as parameters.

A hypercomplex number system E in the n units $e_1, \cdots, e_n$ is said to be reducible when the system satisfies the following conditions†:

$A1$) $E_1 \equiv e_1 \cdots e_m$ forms a system by itself; i. e.,

$$e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3} \qquad (\gamma_{j_1 j_2 k} = 0).$$

$A2$) $E_2 \equiv e_{m+1} \cdots e_n$ forms a system by itself; i. e.,

$$e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \qquad (\gamma_{k_1 k_2 j} = 0).$$

$B1$) $$e_j e_k = 0 \qquad (\gamma_{jkl} = 0),$$

$B2$) $$e_k e_j = 0 \qquad (\gamma_{kjl} = 0);$$

otherwise the system is said to be irreducible.‡

Under these conditions the group G (2) is evidently the direct product of two groups. Two general numbers of the system

$$E \equiv E_1 E_2 \equiv e_1 \cdots e_m e_{m+1} \cdots e_n$$

are

$$x = \sum_{j_1} x_{j_1} e_{j_1} + \sum_{k_1} x_{k_1} e_{k_1}, \qquad y = \sum_{j_2} y_{j_2} e_{j_2} + \sum_{k_2} y_{k_2} e_{k_2};$$

their product is

$$x' = xy = \sum_{j_1} \sum_{j_2} x_{j_1} y_{j_2} e_{j_1} e_{j_2} + \sum_{j_1} \sum_{k_2} x_{j_1} y_{k_2} e_{j_1} e_{k_2} + \sum_{k_1} \sum_{j_2} x_{k_1} y_{j_2} e_{k_1} e_{j_2} + \sum_{k_1} \sum_{k_2} x_{k_1} y_{k_2} e_{k_1} e_{k_2}. \tag{3}$$

* For the conditions imposed on the constants $\gamma_{i_1 i_2 i_3}$ as well as for other details concerning the connection between linear homogeneous groups and hypercomplex number systems see LIE-SCHEFFERS, *Continuierliche Gruppen*, Chapter 21.

† The terms *reducible* and *irreducible* are due to SCHEFFERS, Mathematische Annalen, vol. 39 (1891), p. 317; *Continuierliche Gruppen*, p. 660. As Dr. H. E. HAWKES pointed out in his paper, *Estimate of Peirce's Linear Associative Algebra*, American Journal of Mathematics, vol. 24 (1902), p. 92, these are merely new names for PEIRCE's "mixed" and "pure" systems, ibid., vol. 4 (1881), p. 100.

‡ This is meant in the sense of equivalence; if the units of a given system do not fulfill all of these conditions, but if certain linear combinations with constant coefficients of the units do, then the system will still be called reducible. On the other hand if the given units do not fulfill all of these conditions, nor do any linear combinations with constant coefficients do so either, then the system is irreducible.

According to $(A1)$, $(A2)$, $(B1)$, $(B2)$ we have

$$x' = xy = \sum_{j_1}\sum_{j_2}\sum_{j_3}\gamma_{j_1 j_2 j_3} y_{j_2} x_{j_1} e_{j_3} + \sum_{k_1}\sum_{k_2}\sum_{k_3}\gamma_{k_1 k_2 k_3} y_{k_2} x_{k_1} e_{k_3}.$$

But

$$x' = xy = \sum_{j_3} x'_{j_3} e_{j_3} + \sum_{k_3} x'_{k_3} e_{k_3},$$

and therefore

$$(4)\qquad \begin{aligned} x'_{j_3} &= \sum_{j_1}\sum_{j_2}\gamma_{j_1 j_2 j_3} y_{j_2} x_{j_1} && (j_3 = 1, \cdots, m),\\ x'_{k_3} &= \sum_{k_1}\sum_{k_2}\gamma_{k_1 k_2 k_3} y_{k_2} x_{k_1} && (k_3 = m+1, \cdots, n). \end{aligned}$$

The group (4) may be written schematically

$$(4')\qquad \left|\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1m} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & a_{m+1\,m+1} & \cdots & a_{m+1\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{n\,m+1} & \cdots & a_{nn} \end{array}\right| \quad \text{or} \quad \left|\begin{array}{c|c} G_{11} & 0 \\ \hline 0 & G_{22} \end{array}\right|.$$

A group is said to be reducible* when it can be put in the form

$$(5)\qquad \left|\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1m} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} & 0 & \cdots & 0 \\ \hline a_{m+1\,1} & \cdots & a_{m+1\,m} & a_{m+1\,m+1} & \cdots & a_{m+1\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} & a_{n\,m+1} & \cdots & a_{nn} \end{array}\right| \quad \text{or} \quad \left|\begin{array}{c|c} G_{11} & 0 \\ \hline G_{21} & G_{22} \end{array}\right|.$$

It is therefore evident that under the PEIRCE-SCHEFFERS requirements $A1$, $A2$, $B1$, $B2$, the group of the system is the direct product of G_{11} by G_{22}, where G_{11} is the group of E_1 and G_{22} is the group of E_2.

Retaining the conditions $A1$, $A2$, but replacing $B1$, $B2$, by the less exacting conditions:

$$C1)\qquad e_j e_k = \sum_{k_3}\gamma_{jkk_3} e_{k_3}, \qquad (\gamma_{jkk_3} = 0),$$

$$C2)\qquad e_k e_j = \sum_{k_3}\gamma_{kjk_3} e_{k_3}, \qquad (\gamma_{kjk_3} = 0),$$

* LOEWY, Transactions, vol. 4, January, 1903. The present paper is the outcome of a question which Dr. H. E. HAWKES of Yale University once proposed to me in a conversation, viz., "What will be true of the number system when its group is reducible, i. e., has the form (5)"?

we obtain, by substituting in (3),

$$x' = xy = \sum_{j_1}\sum_{j_2}\sum_{j_3}\gamma_{j_1 j_2 j_3}x_{j_1}y_{j_2}e_{j_3} + \sum_{j_1}\sum_{k_2}\sum_{k_3}\gamma_{j_1 k_2 k_3}x_{j_1}y_{k_2}e_{k_3}$$

(6)
$$+\sum_{k_1}\sum_{j_2}\sum_{k_3}\gamma_{k_1 j_2 k_3}x_{k_1}y_{j_2}e_{k_3} + \sum_{k_1}\sum_{k_2}\sum_{k_3}\gamma_{k_1 k_2 k_3}x_{k_1}y_{k_2}e_{k_3}.$$

But

$$x' = xy = \sum_{j_3} x'_{j_3}e_{j_3} + \sum_{k_3} x'_{k_3}e_{k_3},$$

and therefore,

$$x'_{j_3} = \sum_{j_1}\sum_{j_2}\gamma_{j_1 j_2 j_3}y_{j_2}x_{j_1}$$

(7)
$$x'_{k_3} = \sum_{j_1}\sum_{k_2}\gamma_{j_1 k_2 k_3}y_{k_2}x_{j_1} + \sum_{k_1}\sum_{j_2}\gamma_{k_1 j_2 k_3}y_{j_2}x_{k_1} + \sum_{k_1}\sum_{k_2}\gamma_{k_1 k_2 k_3}y_{k_2}x_{k_1}$$

$$(j_3 = 1, \cdots, m\,;\ k_3 = m+1, \cdots, n).$$

The group (7) has the form

$$\begin{array}{c|c} G_{11} & 0 \\ \hline G_{21} & G_{22} \end{array},$$

where G_{22} is

(8)
$$x'_{k_3} = \sum_{k_1}\Big\{\sum_{j_2}\gamma_{k_1 j_2 k_3}y_{j_2} + \sum_{k_2}\gamma_{k_1 k_2 k_3}y_{k_2}\Big\}x_{k_1} \qquad (k_3 = 1, \cdots, m).$$

The group of E_2 is

(9)
$$x'_{k_3} = \sum_{k_1}\sum_{k_2}\gamma_{k_1 k_2 k_3}y_{k_2}x_{k_1},$$

and thus we see that while G_{11} is the group of E_1, G_{22} is not the group of E_2 unless the $\gamma_{k_1 j_2 k_3} = 0$.

It seems advisable to call the number system E *semireducible of the first kind* when the conditions $A1$, $A2$, $C1$, $C2$ are fulfilled; E is said to be reducible when the conditions $A1$, $A2$, $B1$, $B2$ are satisfied. Clearly the latter is a special case of the former.

As an example we may consider the one given by STUDY *

(10)
$$E \equiv E_1 E_2 \equiv (e_1 e_2)(e_3)$$

whose multiplication table is

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_1	e_3
e_3	e_3	$-e_3$	0

*STUDY, Monatshefte für Mathematik und Physik, vol. 1 (1890), pp. 296, 336.

and whose group is

$$\begin{aligned} x_1' &= y_1 x_1 + y_2 x_2, \\ x_2' &= y_2 x_1 + y_1 x_2, \\ x_3' &= y_3 x_1 + y_3 x_2 + (y_1 - y_2) x_3. \end{aligned} \tag{11}$$

As the general theory requires,

$$\begin{aligned} x_1' &= y_1 x_1 + y_2 x_2, \\ x_2' &= y_2 x_1 + y_1 x_2, \end{aligned}$$

is the group of the system $E_1 \equiv e_1 e_2$.

According to PEIRCE-SCHEFFERS the system E is not reducible since the conditions $B1$ and $B2$ are not satisfied. The system (10) is semireducible of the first kind.

§ 2.

In § 1 it was shown that if the system

$$E = E_1 E_2$$

is semireducible of the first kind its group G is reducible;

$$G \equiv \left| \begin{array}{c|c} G_{11} & 0 \\ \hline G_{21} & G_{22} \end{array} \right|; \tag{7}$$

G_{11} is the group of E_1 but G_{22} may or may not be the group of E_2. In the more general case where G_{22} is not the group of E_2 it is nevertheless highly probable that there always exists a system Q which has G_{22} for its group. The system Q may be called the quotient E/E_1 of the system E, E_1 since its group G_{22} is the quotient $G/_{11}$ of the groups G, G_{11}.* In this paper we shall study the special case where the quotient system $Q = E_2$, i. e., where G_{22} is the group of E_2 and the above existence theorem will therefore not be needed. A simple example will suffice at this point to illustrate the more general case. In the example (§ 1, 10, 11) let $y_1 - y_2 = a$, then

$$G_{22} \equiv x_3' = a x_3$$

is clearly not the group of the system $E_2 = e_3$ $(e_3^2 = 0)$. There exists however a system

* G_{22} is called the quotient of G by G_{11} for the following reasons: If the linear differential equation $P_n y = 0$ is reducible: $P_n = P_{n-m} P_m$; LOEWY showed (Leipziger Berichte, Jan., 1902) that the group of $P_n y = 0$ has the form (7) where G_{11} is the group of P_m and G_{22} is the group of P_{n-m},

$$P_n = P_{n-m} P_m, \qquad G = G_{22} G_{11}.$$

Therefore we may regard G as the product of G_{22} by G_{11} just as we regard P_n as the product of P_{n-m} by P_m.

$$Q \equiv q$$

which has G_{22} for its group, namely, the system whose multiplication table is

	q
q	q

Although the following table anticipates § 3 to some extent, it will be easily intelligible.

Products of Units.	Conditions on γ's.	Conditions on Number System.	Name of System.	Group.
$(A1)\ e_{j_1}e_{j_2}=\sum_{j_3}\gamma_{j_1j_2j_3}e_{j_3}$	$\gamma_{j_1j_2k}=0$	$A1, A2, C1, C2$	Irreducible but semireducible of the first kind.	G is reducible, G_{22} is not necessarily the group of E_2.
$(A2)\ e_{k_1}e_{k_3}=\sum_{k_2}\gamma_{k_1k_2k_3}e_{k_3}$	$\gamma_{k_1k_2j}=0$			
$(B1)\ e_j e_k = 0$	$\gamma_{jki}=0$	$A1, A2, C1, B2$	Irreducible but semireducible of the second kind.	G is reducible, G_{22} is the group of E_2.
$(B2)\ e_k e_j = 0$	$\gamma_{kji}=0$	$A1, A2, B1, B2$	Reducible.	G is reducible and is direct product of G_{22} and G_{11}.
$(C1)\ e_j e_k=\sum_{k_3}\gamma_{jkk_3}e_{k_3}$	$\gamma_{jkj_3}=0$			
$(C2)\ e_k e_j=\sum_{k_3}\gamma_{kjk_3}e_{k_3}$	$\gamma_{kjj_3}=0$		Absolutely irreducible.	G is irreducible.

§ 3.

In accordance with the above table the conditions

$$A1,\quad A2,\quad C1,\quad B2$$

are less exacting than those of PEIRCE-SCHEFFERS but more exacting than $A1, A2, C1, C2$; systems satisfying them will be called *semireducible of the second kind.* Semireducibility of the second kind is a special case of semireducibility of the first kind and reducibility is a special case of semireducibility of the second kind.

A system will be said to be *absolutely irreducible* when its group is irreducible. *

*[August 26, 1903. As initially given the definition of absolute irreducibility (cf. also § 2 table) was erroneous; the correction involved an important change in interpretation in § 3. I am indebted to Professor E. H. MOORE for calling my attention to this. Much of the paper was rewritten as a result of his friendly criticism.]

When the system is semireducible of the second kind, the condition $B2$ makes every $\gamma_{k_1 j_2 k_3}$ vanish and consequently, G_{22} (8) and the group (9) become the same. *In semireducible number systems of the second kind, G_{11} is the group of E_1 and G_{22} is the group of E_2.* Or, *when conditions $A1$, $A2$, $C1$, $B2$ are satisfied, then $E/E_1 = E_2$.*

The necessary and sufficient conditions that a number system shall be semireducible of the second kind are that its group G shall be reducible, that G_{11} shall be the group of E_1 and G_{22} the group of E_2.

We proceed to consider the cases in which the system

$$(12)\quad E = E_1 E_2 \cdots E_{p-1} E_p \equiv e_1 \cdots e_{m_1} e_{m_1+1} \cdots e_{m_2} \cdots e_{m_{p-2}+1} \cdots e_{m_{p-1}} e_{m_{p-1}+1} \cdots e_n$$

has the following properties: by a suitable choice of the units it is possible to decompose E in such a way that

$$E_1 \cdots E_h E_{h+1} \qquad (h = 1, \cdots, p-1)$$

is semireducible of the second kind, E_{h+1} being absolutely irreducible. By a linear transformation

$$(13)\quad \bar{e}_{i_1} = \sum_{i_2} \alpha_{i_1 i_2} e_{i_2}, \qquad |\alpha_{i_1 i_2}| \neq 0,$$

the system $E = E_1 \cdots E_p$ may be similarly decomposed in the form $\bar{E} \equiv \bar{E}_1 \cdots \bar{E}_q$. The systems E and $\bar{E}$ are said to be of the same type * and are not regarded as essentially different.

Let the group of E be G and that of $\bar{E}$ be $\bar{G}$. Since E is semireducible of the second kind G has the form

$$G \equiv \begin{matrix} G_{11} & 0 & 0 & \cdots & 0 \\ G_{21} & G_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ G_{p1} & G_{p2} & G_{p3} & \cdots & G_{pp} \end{matrix}$$

where G_{11} is the group of E_1, G_{22} is the group of E_2, $\cdots$, G_{pp} is the group of E_p.

The group of $\bar{G}$ of $\bar{E}$ is obtained from G by a transformation of variables.† G and $\bar{G}$ are thus similar, $\bar{G}$ has the form

$$\bar{G} \equiv \begin{matrix} \bar{G}_{11} & 0 & 0 & \cdots & 0 \\ \bar{G}_{21} & \bar{G}_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{G}_{q1} & \bar{G}_{q2} & G_{q3} & \cdots & G_{qq} \end{matrix}$$

* *Continuierliche Gruppen*, pp. 642–3.

† Loc. cit., p. 643.

According to LOEWY's fundamental theorem * $q = p$ and

$$G_{11},\ G_{22},\ \cdots,\ G_{pp}$$

are similar in some order to

$$\overline{G}_{11},\ \overline{G}_{22},\ \cdots,\ \overline{G}_{pp};$$

let us say that $G_{\alpha\alpha}$ is similar to $\overline{G}_{\beta_\alpha\beta_\alpha}$ $(\alpha = 1, \cdots, p)$.

Thus it is evident that $\overline{E}$ *decomposes into the same number of subsystems as* $E(q = p)$. Furthermore, owing to the similarity of $G_{\alpha\alpha}$ with $\overline{G}_{\beta_\alpha\beta_\alpha}$, the subsystems E_α and $\overline{E}_\beta$ must be similar $(\alpha = 1, \cdots, p)$. Hence, in analogy with LOEWY's theorem for reducible linear homogeneous groups, we have proved for number systems the following theorem:

If a system E *is, according to two different choices of the units, decomposable once into the sequence of* p *absolutely irreducible systems* $E_1, \cdots, E_p$, *where the system*

$$E_1 \cdots E_h E_{h+1}$$

is semireducible of the second kind $(h = 1, 2, \cdots, p-1)$, *and again similarly decomposable into the sequence of* q *systems* $\overline{E}_1 \cdots \overline{E}_q$, *then* $q = p$ *and the subsystems* $\overline{E}_1, \cdots, \overline{E}_q$ *are similar to the subsystems* $E_1, \cdots, E_p$, *apart from the order.* In other words, *similar systems being regarded as not different, the absolutely irreducible subsystems obtained by a linear transformation* (13) *are equivalent, apart from the arrangement, to the subsystems obtained by any other such transformation.*

The preceding theorem includes, of course, the particular case in which semireducibility of the second kind is the reducibility in the sense of PEIRCE-SCHEFFERS. Then the group is the direct product of the groups of the component systems.

THE UNIVERSITY OF CHICAGO,
June, 1903.

* LOEWY, Transactions, vol. 4, January, 1903, pp. 46–47.

A SYMBOLIC TREATMENT OF THE THEORY OF INVARIANTS OF QUADRATIC DIFFERENTIAL QUANTICS OF n VARIABLES*

BY

HEINRICH MASCHKE

In the article † *A new method of determining the differential parameters and invariants of quadratic differential quantics* I have shown that the application of a certain symbolic method leads very readily to the formation of expressions remaining invariant with respect to the transformation of quadratic differential quantics. The presentation in that article was only a preliminary one and the work practically confined to the case of two independent variables. In my paper ‡ *Invariants and covariants of quadratic differential quantics of n variables* a more complete treatment was intended and the investigation applied throughout to the case of n variables, leaving aside, however, simultaneous invariant forms of more than one quantic.

The present paper contains in §§ 1–6 and § 8 essentially the content of the last mentioned paper; the greater parts of § 5 and § 8, and all the remaining articles are new, in particular the extensive use of covariantive differentiation.

§ 1. *Definitions. The fundamental theorem.*

To the given quadratic differential quantic

$$A = \sum_{i,k=1}^{n} a_{ik}\, dx_i\, dx_k, \tag{1}$$

with $x_1, x_2, \cdots, x_n$ as independent variables, and the a_{ik} ($a_{ki} = a_{ik}$) as functions of these variables, we apply the transformation

$$x_i = x_i(y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n), \tag{2}$$

and obtain

$$A = A' = \sum_{i,k=1}^{n} a'_{ik}\, dy_i\, dy_k. \tag{3}$$

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* Presented to the Society (Chicago) April 11, 1903, under the title *Invariants and covariants of quadratic differential quantics of n variables.* Received for publication June 20, 1903.

† Transactions of the American Mathematical Society, vol. 1 (1900), pp. 197–204.

‡ The Decennial Publications of the University of Chicago, Chicago, 1903.

Since the differentials dy are connected linearly with the differentials dx by means of the formulas

$$dx_i = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} dy_k \qquad (i=1, 2, \cdots, n), \tag{4}$$

we have at once, denoting the discriminant of A by $|a_{ik}|$,

$$|a'_{ik}| = r^2 |a_{ik}|, \tag{5}$$

where r denotes the determinant of the linear substitution, viz.:

$$r = \left| \frac{\partial x_i}{\partial y_k} \right|. \tag{6}$$

Let now $u, v, \cdots$, be any (arbitrary) functions of $x_1, x_2, \cdots, x_n$; $u', v', \cdots$ the same functions after application of the transformation (2), then we call every function of the coefficients a_{ik} and their derivatives, and of $u, v, \cdots$ and their derivatives, an *invariant expression* of the quadratic differential quantic A, if the expression remains the same, whether formed with the old quantities $a_{ik}, u, v, \cdots$ and their derivatives with respect to x, or with the new quantities $a'_{ik}, u', v', \cdots$ and their derivatives with respect to y.

From $u' = u$ it follows, for instance, that every arbitrary function of x is an invariant expression of A.

If such an invariant expression involves one or more arbitrary functions $u, v, \cdots$ and their derivatives, it is called a *differential parameter;* if it involves no such functions, if it is therefore a function of the a_{ik} and their derivatives alone, it is called an *invariant proper.* *

By the *order* of an invariant expression we shall understand the order of the highest derivative appearing in it.

Suppose now $F^1, F^2, \cdots, F^n$ are any n invariant expressions of A, then we have

$$F^{i'} = F^i \qquad (i=1, 2, \cdots, n),$$

and also

$$\sum_{k=1}^{n} \frac{\partial F^{i'}}{\partial y_k} dy_k = \sum_{k=1}^{n} \frac{\partial F^i}{\partial x_k} dx_k \qquad (i=1, 2, \cdots, n).$$

It follows now at once that

$$\left| \frac{\partial F^{i'}}{\partial y_k} \right| = r \cdot \left| \frac{\partial F^i}{\partial x_k} \right|,$$

and therefore with reference to (5)

$$|a'_{ik}|^{-\frac{1}{2}} \cdot \left| \frac{\partial F^{i'}}{\partial y_k} \right| = |a_{ik}|^{-\frac{1}{2}} \cdot \left| \frac{\partial F^i}{\partial x_k} \right|. \tag{7}$$

* Cf. LUIGI BIANCHI, *Vorlesungen über Differentialgeometrie;* autorisierte deutsche Uebersetzung von MAX LUKAT, Leipzig (1899), p. 39.

This equation defines the last expression as an invariant expression, and so we have the fundamental theorem:

"*If* $F^1, F^2, \cdots, F^n$ *are any* n *invariant expressions of* A, *then*

$$|a_{ik}|^{-\frac{1}{2}} \cdot \left| \frac{\partial F^i}{\partial x_k} \right|$$

is again an invariant expression of A."

§ 2. *Choice of convenient notations.*

Since we shall have in the following to compute continuously with expressions of the type (7), a shorter notation is indispensable. Let us first agree to indicate differentiation by subscripts. F being any quantity whatever we write

$$\frac{\partial F}{\partial x_i} = F_i. \tag{8}$$

We further denote the reciprocal value of the (positive) square root of the discriminant of A — which we always suppose to be different from zero — by the single letter β:*

$$\beta = |a_{ik}|^{-\frac{1}{2}}. \tag{9}$$

The functional determinant of any n quantities F^i $(i = 1, 2, \cdots, n)$ will be denoted by

$$\{F^1, F^2, F^3, \cdots, F^n\},$$

so that we have

$$\{F^1, F^2, F^3, \cdots, F^n\} = \left| \frac{\partial F^i}{\partial x_k} \right|. \tag{10}$$

The product of β into such a functional determinant will be denoted by

$$(F^1, F^2, \cdots, F^n) = \beta \{F^1, F^2, \cdots, F^n\}. \tag{11}$$

The quantities we have almost exclusively to deal with in the sequel are not the functional determinants themselves, but their products into β, and for this reason we use for the latter quantities the simpler symbol ( ) instead of { }.†

Even this notation is in most cases too cumbersome. We write then simply

$$(F^1, F^2, \cdots, F^n) = (F). \tag{12}$$

If it should be necessary to indicate the first, or the first two, three, etc., quantities of such a parenthesis distinctly, we write them in their proper places and let the last letter run out. For instance

* α in my previous paper.

† I therefore withdraw the suggestion made in my previous paper, loc. cit., p. 199, footnote, where the two parentheses ( ) and { } were used in the reversed sense.

$$(a) \text{ means } (a^1, a^2, \cdots, a^n),$$

(13) $$(b, a) \text{ means } (b, a^2, a^3, \cdots, a^n),$$

$$(b, c, a) \text{ means } (b, c, a^3, a^4, \cdots, a^n).$$

It is understood that the letter a occurring in the coefficients a_{ik} of A has no connection whatever with the letter a occurring, for instance, in (b, c, a).

The quantities in a parenthesis ( ) should be separated by commas. If, however, no misrepresentation can occur, the commas may be omitted:

$$(bca) = (b, c, a), \text{ etc.}$$

In our new notation the last theorem of § 1 is now this: "*If* $F^1, F^2, \cdots, F^n$ *are invariant expressions of* A, *then* (F) *is also an invariant expression of* A;" we shall call it an *invariantive constituent.*

§ 3. *The symbolic method.*

If we define

(14) $$f_i f_k = a_{ik},$$

we have

(15) $$A = \sum_{ik} a_{ik} dx_i dx_k = [\sum_i f_i dx_i]^2.$$

The expression $\sum_i f_i dx_i$ appears, if we think of the notation $f_i = \partial f / \partial x_i$ agreed upon in (8) as the complete differential of a (symbolic) function of the n variables x.

If expressions of higher than the first dimensions in the coefficients a_{ik} are to be formed symbolically, we have to use different symbols $f, \phi, \cdots$.

For instance

$$a_{12}^2 = f_1 f_2 \phi_1 \phi_2, \qquad a_{11} a_{22} = f_1 f_1 \phi_2 \phi_2, \text{ etc.}$$

The symbolic functions $f, \phi, \cdots$ (we shall simply call them symbols) appear now, as every arbitrary function $u, v, \cdots$ of the variables x does, as invariant expressions of A.

If now we form invariantive constituents containing the symbols $f, \phi, \cdots$ or $f^1, f^2, \cdots$ and any number of arbitrary functions $u, v, \cdots$, then every product of these constituents will represent an expression which, according to the fundamental theorem of § 1, will represent an expression which is *formally* invariant. But from the same reasoning as in algebra it follows that these products will at once represent *actual* invariant expressions, as soon as the symbols $f, \phi, \cdots$ occur in such connections as to permit actual meaning, e. g., in the connection $f_i f_k, \phi_i \phi_k$, etc. The connections of this type, however, are not the only ones.

Some of the elements of the different invariantive constituents may be constituents themselves, as for instance in the examples

$$\big((fa), b\big)\cdot(fc) \qquad \text{or} \qquad \big((fa), (fb), c\big),$$

where f denotes a symbol. In such cases also the higher derivatives of the symbol f will occur, and it is then the question, whether or not these combinations have actual meaning (they do in the above examples). Thus we have the following theorem:

Every product of invariantive constituents, the elements of which are symbols or arbitrary functions or both, or again invariantive constituents of the same character, represents an invariant expression of A, provided that every symbol occurs precisely twice and in such a connection as to permit actual interpretation in terms of the a_{ik} and their derivatives.

With regard to these symbolic invariant expressions the following two theorems are evidently true:

The value of an invariant expression in symbolic form is not changed if two equivalent symbols are interchanged, and:

An invariant expression in symbolic form vanishes if by the interchange of two equivalent symbols its sign is changed.

Covariants can now also be formed easily. In the first place, the complete differential of every invariant expression represents immediately a linear covariant of A. Let us denote for simplicity

$$F_1 dx_1 + F_2 dx_2 + \cdots F_n dx_n = F_x. \tag{16}$$

A single symbol and also any invariantive constituent with the subscript x represents then, at least formally, a linear covariant — let us call it a *covariantive constituent.* We therefore obtain covariants of any degree by forming products of these covariantive and invariantive constituents. Thus we have the theorem:

Every product of covariantive and invariantive constituents represents an actual covariant of A, provided that every symbol occurs precisely twice and in such a connection as to permit actual interpretation in terms of the coefficients a_{ik} and their derivatives. The degree of the covariant is determined by the number of the covariantive constituents as factors in the product.

§ 4. *Some important invariant expressions.*

The simplest possible invariant proper is $(f)^2$, with $f^1, f^2, \cdots f^n$ as equivalent symbols. But this invariant reduces to a constant. To show it we compute first the product

$$P = f^1_1 f^2_2 \cdots f^n_n \{f^1, f^2, \cdots, f^n\}.$$

If we multiply the first row of the functional determinant by f_1^1, the second by f_2^2, etc., we find at once

$$P = |a_{ik}| = \beta^{-2}.$$

If, on the other hand, we permute in P the equivalent symbols f^i in all possible ways, and add the products so obtained, which are all equal, together, we have

$$n!\,P = \{f^1, f^2, \cdots, f^n\}^2.$$

Since now

$$(f)^2 = \beta^2 \{f^1, f^2, \cdots, f^n\}^2,$$

we have the important formula

$$(f)^2 = n!. \tag{17}$$

Let us next compute the invariant expression $(uf)^2$, where u denotes an arbitrary function and $f^2, f^3, \cdots, f^n$ as before, equivalent symbols. For this purpose, and also for the following, we need a proper notation for the minors of some of our determinants. We denote

(18) by A_{ik}* the minor of the element a_{ik} in the determinant $|a_{ik}|$,

and further

(19) by $F^{i,k}$ the minor of the element f_k^i, in the determinant $\{f^1, f^2, \cdots, f^n\}$.

In this notation we have

$$(uf)^2 = \beta^2 \sum_{ik} F^{1,i} \cdot F^{1,k} u_i u_k.$$

But

$$F^{1,i}F^{1,k} = (n-1)!\,f_1 f_2 \cdots f_{i-1} f_{i+1} \cdots f_n \cdot \begin{vmatrix} f_1^2, f_2^2, \cdots, f_{k-1}^2, f_{k+1}^2, \cdots, f_n^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ f_1^n, f_2^n, \cdots, f_{k-1}^n, f_{k+1}^n, \cdots, f_n^n \end{vmatrix},$$

or

$$F^{1,i}F^{1,k} = (n-1)!\,A_{ik},$$

and also

$$F^{\lambda,i}F^{\lambda,k} = (n-1)!\,A_{ik}. \tag{20}$$

Hence

$$\frac{1}{(n-1)!}(uf)^2 = \beta^2 \sum_{ik} A_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k}, \dagger \tag{21}$$

which is the well-known first differential parameter $\Delta_1 u$. Thus we have

$$(uf)^2 = (n-1)!\,\Delta_1 u. \tag{22}$$

* BIANCHI's quantities A_{ik} (cf. loc. cit., p. 37) have a slightly different signification. His A_{ik} is in our notation $= \beta^2 A_{ik}$.

† BIANCHI, loc. cit., p. 41.

In the same way we obtain

$$\frac{1}{(n-1)!}(uf)(vf) = \beta^2 \sum_{ik} A_{ik} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_k},$$ *

or

$$(uf)(vf) = (n-1)!\,\Delta(u, v). \tag{23}$$

Numerous other differential parameters can be formed: e. g. $(uvf)^2$, etc.

In all these examples of invariant expressions only the coefficients a_{ik} themselves occur, not their derivatives. In order to derive invariant expressions involving also derivatives of the a_{ik}, we have to express these in terms of our symbols.

We deduce from (14) by differentiation

$$\begin{aligned} \frac{\partial a_{ik}}{\partial x_l} &= f_i f_{kl} + f_k f_{il}, \\ \frac{\partial a_{kl}}{\partial x_i} &= f_k f_{li} + f_l f_{ki}, \\ \frac{\partial a_{li}}{\partial x_k} &= f_l f_{ik} + f_i f_{lk}, \end{aligned} \tag{24}$$

which gives at once

$$f_i f_{kl} = \tfrac{1}{2}\left[\frac{\partial a_{ik}}{\partial x_l} + \frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{kl}}{\partial x_i}\right]. \tag{25}$$

The expression on the right side is precisely CHRISTOFFEL's † so-called triple index symbol denoted briefly by $\left[\begin{matrix} kl \\ i \end{matrix}\right]$. ‡

We have thus the important theorem: *The symbolic product $f_i f_{kl}$ has actual meaning for every system of values i, k, l; it is equal to the triple index symbol*

$$f_i f_{kl} = \left[\begin{matrix} kl \\ i \end{matrix}\right]. \tag{26}$$

We see, then, further, that the following combinations permit actual interpretation in the second derivatives of the a_{ik}:

$$f_{im} f_{kl} + f_i f_{klm} = \left[\begin{matrix} kl \\ i \end{matrix}\right]_m,$$

* BIANCHI, loc. cit., p. 41.

† CHRISTOFFEL, *Ueber die Transformation der homogenen Differentialausdrücke des zweiten Grades*, Crelle's Journal, vol. 70, p. 48.

‡ BIANCHI, loc. cit., p. 43.

$$f_{ir}f_{ks}-f_{kr}f_{is}=\begin{bmatrix} ir \\ k \end{bmatrix}_s-\begin{bmatrix} is \\ k \end{bmatrix}_r, \tag{27}$$

$$f_{rs}f_{ikn}-f_{sn}f_{ikr}=\begin{bmatrix} ik \\ r \end{bmatrix}_s-\begin{bmatrix} ik \\ s \end{bmatrix}_r.$$

The simplest symbolic invariant expression involving the second derivatives of the symbols f is $((uf),f)$. The computation, which will be given at the end of § 6, leads to the result

$$\frac{1}{(n-1)!}((uf),f)=\beta^2\cdot\sum_{ik}A_{ik}\frac{\partial^2 u}{\partial x_i\partial x_k}-\beta^1\cdot\sum_{iklm}{}'A_{ik}A_{lm}\begin{bmatrix} ik \\ l \end{bmatrix}\frac{\partial u}{\partial x_m}.$$

This expression is the second differential parameter $\Delta_2 u$.* Hence we have

$$((uf),f)=(n-1)!\,\Delta_2 u. \tag{28}$$

§ 5. *Relations between symbolic expressions.*

For our further computation with symbolic expressions it is necessary to deduce a number of fundamental relations. As a matter of convenience (not of definition) we shall as a rule denote symbols by the letters $a, b, c, \cdots, f, \phi, \cdots$, and any functions of the x (not necessarily symbols) by the letters $u, v, \cdots$ and $x, y, \cdots$.

Differentiating formula (17) with regard to x_i, we have the fundamental formula

$$(f)(f)_i=0. \tag{29}$$

Let now

(30) $[f]$ stand for any alternating function of $f^1, f^2, \cdots, f^n$,

e. g., for (f) or for $(f)_i$ or any higher ordinary or covariantive (see § 7) derivative of (f), and let us form the symbolic product

$$f_1^1(uf)[f].$$

We find

$$f_1^1\{uf\}[f]=\begin{vmatrix} f_1^1u_1, & u_2, & \cdots, & u_n \\ f_1^1f_1^2, & f_2^2, & \cdots, & f_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ f_1^1f_1^n, & f_2^n, & \cdots, & f_n^n \end{vmatrix}\cdot[f]=\begin{vmatrix} f_1^1u_1, & u_2, & \cdots, & u_n \\ 0, & f_2^2, & \cdots, & f_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & f_2^i, & \cdots, & f_n^n \end{vmatrix}\cdot[f]$$

because the product $f_1^1f_1^i[f]$ changes its sign if the equivalent symbols f^1 and f^i are interchanged, and must therefore vanish; i. e.,

$$f_1^1f_1^i[f]=0.$$

* BIANCHI, loc. cit., p. 47.

It now follows further that

$$f_1^1\{uf\}[f] = (n-1)!\,u_1 f_1^1 f_2^2 \cdots f_n^n[f] = \frac{(n-1)!}{n!}u_1\{f\}[f],$$

and

$$f_1^1(uf)[f] = \frac{1}{n}u_1(f)[f].$$

The same method can be applied when we operate with f_i^1 instead of f_1^1, so that we also have

$$f_i^1(uf)[f] = \frac{1}{n}u_i(f)[f],$$

or, changing the notation

$$f_i(ua)[fa] = \frac{1}{n}u_i(fa)[fa]. \tag{31}$$

If now we specify the symbol $[f]$ according to (30) we obtain the two equations

$$f_k(fa)(ua) = (n-1)!\,u_k, \tag{32}$$

$$f_k(fa)_i(ua) = 0. \tag{33}$$

With $v^2, v^3, \cdots, v^n$ as arbitrary functions these formulas can at once be extended to

$$(fa)(fv)(ua) = (n-1)!(uv), \tag{34}$$

$$(fa)_i(fv)(ua) = 0. \tag{35}$$

A similar method serves to reduce the expression

$$f_1^1 f_2^2(uvf)[f].$$

We find

$$f_1^1 f_2^2\{uvf\}[f] = \begin{vmatrix} f_1^1 u_1, & f_2^2 u_2, & u_3, & \cdots, & u_n \\ f_1^1 v_1, & f_2^2 v_2, & v_3, & \cdots, & v_n \\ f_1^1 f_1^3, & f_2^2 f_2^3, & f_3^3, & \cdots, & f_n^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f_1^1 f_1^n, & f_2^2 f_2^n, & f_3^n, & \cdots, & f_n^n \end{vmatrix} \cdot [f]$$

$$= \begin{vmatrix} f_1^1 u_1, & f_2^2 u_2, & u_3, & \cdots, & u_n \\ f_1^1 v_1, & f_2^2 v_2, & v_3, & \cdots, & v_n \\ 0, & 0, & f_3^3, & \cdots, & f_n^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & f_3^n, & \cdots, & f_n^n \end{vmatrix} \cdot [f].$$

Further

$$f_1^1 f_2^2 \{uvf\}[f] = (n-2)!(u_1v_2 - u_2v_1) f_1^1 f_2^2 f_3^3 \cdots f_n^n [f]$$
$$= \frac{(n-2)!}{n!}(u_1v_2 - u_2v_1)\{f\}[f],$$

and changing notation

$$(36) \qquad f_i\phi_k(uva)[f\phi a] = \frac{1}{n(n-1)}(u_iv_k - u_kv_i)(f\phi a)[f\phi a].$$

Specifying again the symbol $[f\phi a]$ we deduce

$$(37) \qquad f_i\phi_k(uva)(f\phi a) = (n-2)!(u_iv_k - u_kv_i),$$

$$(38) \qquad f_i\phi_k(uva)(f\phi a)_i = 0.$$

From the formulas thus obtained a great number of others can be deduced. A number partly to be applied in the following articles are here listed. They involve ϵ, η as abbreviations:

$$(39) \qquad \epsilon = \frac{1}{(n-1)!}, \qquad \eta = \frac{1}{(n-2)!}.$$

$$(40) \quad (fu)(xa)[fa] = \frac{1}{n}(xu)(fa)[fa]. \qquad \text{From (31).}$$

$$(41) \quad [f_{ik}(xa) + f_i(xa)_k][fa] = \frac{1}{n}[x_{ik}(fa) + x_i(fa)_k][fa].$$

From (31) by differentiation, and the observation that also

$$f_i(xa)[fa]_k = \frac{1}{n}x_i(fa)[fa]_k.$$

$$(42) \quad [f_i(xa)_k - f_k(xa)_i][fa] = \frac{1}{n}[x_i(fa)_k - x_k(fa)_i][fa]. \qquad \text{From (41).}$$

$$(43) \quad \big((xa), f, u\big)[fa] = \frac{1}{n}\big((fa), x, u\big)[fa]. \qquad \text{From (42).}$$

$$(44) \quad [(xa)_i(fu) + (xa)(fu)_i][fa]$$
$$= \frac{1}{n}[(xu)_i(fa) + (xa)(fu)_i][fa]. \qquad \text{From (40).}$$

$$(45) \quad [f_i(xa)_k - f_k(xa)_i](fa) = 0. \qquad \text{From (42).}$$

$$(46) \quad \epsilon[(xa)_i(fu) + (xa)(fu)_i](fa) = (xu)_i. \qquad \text{From (44).}$$

$$(47) \quad [(xa)_i(fu) + (xa)(fu)_i](fa)_k = \frac{1}{n}(xu)(fa)_i(fa)_k. \qquad \text{From (44).}$$

(48) $f_i\big((fa), u\big)(xa) = 0.$ From (33).

(49) $(fu)\big((fa), v\big)(xa) = 0.$ From (48).

(50) $\big((fa), f, u\big)(xa) = 0.$ From (33).

(51) $\big((fa), u\big)(fa) = 0.$ From (29).

(52) $\big((xa), f, u\big)(fa) = 0.$ From (43) and (51).

(53) $x_{ik} = \epsilon f_i(fa)(xa)_k + \epsilon f_{ik}(fa)(xa).$ From (41).

(54) $f_i(xa)_k(fa) = f_k(xa)_i(fa).$ From (53).

(55) $f_i(fa)_l(xa)_k = \dfrac{1}{n} x_i(fa)_k(fa)_l - f_{ik}(fa)_l(xa).$ From (41).

(56) $\big((fa), (xa), f, u\big) = 0.$ From (55).

(57) $f_i(\phi u)(xya)[f\phi a] = \dfrac{1}{n(n-1)}[x_i(yu) - y_i(xu)](f\phi a)[f\phi a].$ From (36).

(58) $(f\phi u)(xya)[f\phi a] = \dfrac{2}{n(n-1)}(xyu)(f\phi a)[f\phi a].$ From (36).

(59) $(fu)(\phi v)(xya)[f\phi a]$

$$= \frac{1}{n(n-1)}[(xu)(yv) - (xv)(yu)](f\phi a)[f\phi a].$$ From (36).

(60) $\eta f_i(f\phi a)(\phi u)(xya) = x_i(yu) - y_i(xu).$ From (57).

(61) $\eta(f\phi u)(f\phi a)(xya) = 2(xyu).$ From (58).

(62) $\eta(fu)(\phi v)(f\phi a)(xya) = (xu)(yv) - (xv)(yu).$ From (59).

(63) $f_i(\phi u)(f\phi a)_k(xya) = 0.$ From (57).

(64) $(f\phi u)(f\phi a)_i(xya) = 0.$ From (58).

(65) $(fu)(\phi v)(f\phi a)_i(xya) = 0.$ From (59).

We notice that some of these equations contain in every term connections of the form $f_{ik}f_{rs}$, e. g., equation (55). Indeed the different terms of this and similar equations have no actual meaning. Nevertheless the formulas are formally correct and can be used with safety for reduction work.

§ 6. *The quadratic covariant expression and the triple index symbols of the second kind.*

The quantity $f_i(ua)_k(fa)$ which according to (54) remains unchanged when the two indices i and k are interchanged gives rise to the quadratic covariant expression

$$(fa)f_x(ua)_x = \sum_{ik}(fa)f_i(ua)_k dx_i dx_k, \tag{66}$$

whose coefficients can be shown to be equal to the "*covariant second derivatives of the function u*" in BIANCHI's terminology.*

For this purpose we have first to compute the triple index symbol of the second kind, †

$$\left\{ \begin{matrix} ik \\ l \end{matrix} \right\} = \beta^2 \sum_m A_{lm} \left[\begin{matrix} ik \\ m \end{matrix} \right]. \tag{67}$$

From (20) and (26) we have

$$(n-1)! \left\{ \begin{matrix} ik \\ l \end{matrix} \right\} = \beta^2 f^1_{ik} F^{1,l} \cdot \sum_m f^1_m F^{1,m} = \beta f^1_{ik} F^{1,l} \cdot (f),$$

$$\left\{ \begin{matrix} ik \\ l \end{matrix} \right\} = \epsilon\beta f^1_{ik} F^{1,l} \cdot (f), \tag{68}$$

which is the required symbolic representation.

We derive from this equation

$$\sum_m u_m \left\{ \begin{matrix} ik \\ m \end{matrix} \right\} = \epsilon\beta f^1_{ik}(f) \sum_m u_m F^{1,m},$$

which gives, with a slight change of notation,

$$\sum_m u_m \left\{ \begin{matrix} ik \\ m \end{matrix} \right\} = \epsilon f_{ik}(ua)(fa). \tag{69}$$

Now BIANCHI's covariant second derivatives of u are defined as follows: ‡

$$u^{(ik)} = \frac{\partial^2 u}{\partial x_i \partial x_k} - \sum_m \frac{\partial u}{\partial x_m} \left\{ \begin{matrix} ik \\ m \end{matrix} \right\}. \tag{70}$$

Hence

$$u^{(ik)} = u_{ik} - \epsilon f_{ik}(ua)(fa), \tag{71}$$

and by means of (53)

$$u^{(ik)} = \epsilon f_i(ua)_k(fa), \tag{72}$$

which formula verifies the above statement concerning the coefficients of the quadratic covariant expression (66).

This formula leads now also to the proof of (28).

BIANCHI defines §

* Loc. cit., p. 46.

† CHRISTOFFEL, loc. cit., p. 49, and BIANCHI, loc. cit., p. 43.

‡ Loc. cit., p. 46.

§ Loc. cit., p. 47, formula (24).

$$\Delta_2 u = \beta^2 \sum_{ik} A_{ik} u^{(ik)}, \tag{73}$$

which gives

$$(n-1)!\Delta_2 u = \beta^2 (fa) \sum_{ik} A_{ik} f_i (ua)_k .$$

But from (21)

$$\beta^2 \sum_{ik} A_{ik} f_i (ua)_k = \epsilon (f\phi)\big((ua), \phi\big),$$

and from (34)

$$(fa)(f\phi)\big((ua), \phi\big) = (n-1)!\big((ua), a\big),$$

therefore

$$(n-1)!\Delta_2 u = \big((ua), a\big).$$

§ 7. *Covariantive differentiation.*

For the formation of invariants and covariants, which involve explicitly derivatives higher than the first, it turns out to be of the greatest advantage to use instead of ordinary differentiation another process which might be called *covariantive differentiation.** We shall use for its notation upper indices in parentheses.

Let x stand for any quantity not involving derivatives, then the first covariantive derivative is the same as the first ordinary derivative:

$$x^{(\lambda)} = x_\lambda . \tag{74}$$

The second covariantive derivative, suggested by formula (71) is defined as follows

$$x_i^{(k)} = x^{(ik)} = x_{ik} - \epsilon f_{ik} (fa)(xa). \tag{75}$$

On account of (53) the following definition might also be used

$$x^{(ik)} = \epsilon f_i (xa)_k (fa). \tag{76}$$

We have then from (70)

$$\sum_\lambda x_\lambda \left\{ \begin{matrix} ik \\ \lambda \end{matrix} \right\} = x_{ik} - x^{(ik)}. \tag{76a}$$

It further follows from (54) that

$$x^{(ik)} = x^{(ki)}. \tag{77}$$

If a product is to be differentiated covariantively we apply the same rule as in ordinary differentiation, i. e.,

$$[x_i y_k]^{(\lambda)} = x_i y^{(k\lambda)} + y_k x^{(i\lambda)}. \tag{78}$$

*See the remark on CHRISTOFFEL's process at the introduction to § 9. The same process has been called by RICCI "dérivation covariante;" Bulletin des Sciences Mathématiques, ser. 2, vol. 16 (1892), p. 175.

We now have to determine the covariantive derivative of an invariant constituent (f). In order to avoid however too lengthy formulas we carry the work through at first for the case $n = 3$.

To the terms of the equation

$$\{f\phi\psi\}^{(\Lambda)} = \{f^{(\Lambda)}\phi\psi\} + \{f\phi^{(\Lambda)}\psi\} + \{f\phi\psi^{(\Lambda)}\}$$

we apply (75) so that

$$\{f^{(\Lambda)}\phi\psi\} = \{f_\Lambda\phi\psi\} - \epsilon(\chi a)(fa)\{\chi_\Lambda\phi\psi\}.$$

Hence

$$\{f\phi\psi\}^{(\Lambda)} = \{f\phi\psi\}_\Lambda - \epsilon[(fa)\{\chi_\Lambda\phi\psi\} + (\phi a)\{f\chi_\Lambda\psi\} + (\psi a)\{f\phi\chi_\Lambda\}](\chi a).$$

Using now the identity

$$79) \qquad \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ (au), & (bu), & (cu), & (du) \end{vmatrix} \equiv 0,$$

and setting $a = f$, $b = \phi$, $c = \psi$, $d = \chi_\Lambda$, $u = a$ we reduce the quantity in the bracket to

$$\{f\phi\psi\}(\chi_\Lambda a).$$

But

$$\{\chi a\}\{\chi_\Lambda a\} = \frac{1}{n}\{\chi a\}\{\chi a\}_\Lambda,$$

because χ and every one of the a's are equivalent symbols. With reference to (29) and (17) we find now

$$(\chi_\Lambda a)(\chi a) = -(n-1)!\,\frac{\beta_\Lambda}{\beta},$$

and therefore

$$\beta\{f\phi\psi\}^{(\Lambda)} = \beta\{f\phi\psi\}_\Lambda + \beta_\Lambda\{f\phi\psi\},$$

i. e.,

$$\beta\{f\phi\psi\}^{(\Lambda)} = (f\phi\psi)_\Lambda.$$

For n variables the proof is quite analogous by using the identity

$$(80) \qquad \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 & \cdots & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & \cdots & a_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a^1u) & (a^2u) & (a^3u) & \cdots & (a^nu) \end{vmatrix} \equiv 0.$$

Thus we have also for the general case of n variables the result

$$\beta\{f\}^{(\lambda)} = (f)_\lambda. \tag{81}$$

In order to determine

$$(f)^{(\lambda)} = \beta^{(\lambda)}\{f\} + \beta\{f\}^{(\lambda)}$$

we have to find the value of $\beta^{(\lambda)}$.

The covariantive differentiation of the equation

$$n!\,\beta^{-2} = \{a\}^2$$

leads to

$$-2n!\,\beta^{-3}\beta^{(\lambda)} = 2\{a\}\{a\}^{(\lambda)}.$$

This reduces by means of (79) to

$$\frac{2}{\beta^2}(a)(a)_{(\lambda)} = 0, \text{ i. e.,}$$

$$\beta^{(\lambda)} = 0. \tag{82}$$

With respect to covariantive differentiation the quantity β represents a constant.

Now we have from (81) and (82) the important result

$$(f)^{(\lambda)} = (f)_\lambda. \tag{83}$$

By means of this and the preceding theorems and definitions the third and higher covariantive derivatives can now be formed without any further difficulty.

One of the advantages of computing with covariantive derivatives lies in the fact that the product of a first and a second covariantive derivative of a symbol vanishes.

Indeed from

$$f_k f^{(il)} = \epsilon f_k \phi_i (fa)_l (\phi a)$$

it follows by means of (33) that

$$f_k f^{(il)} = 0, \tag{84}$$

and also

$$(fu) f^{(ik)} = 0. \tag{85}$$

The following formulas can now easily be established:

$$(f)(f)^{(ik)} = (f)(f)_{ik} = -(f)_i (f)_k. \tag{86}$$

$$[f^{(ik)}(xa) + f_i (xa)_k][fa] = \frac{1}{n}[x^{(ik)}(fa) + x_i (fa)_k][fa]. \quad \text{From (31).} \tag{87}$$

$$((xa), f, u)_{(d)}[fa] = \frac{1}{n}((fa), x, u)_{(d)}[fa]. \quad \text{From (43),} \tag{88}$$

where the suffix (d) indicates ordinary or covariantive differentiation with respect to any number of variables.

(89) $[f_i(xa)_k - f_k(xa)_i]_{(d)}[fa] = \frac{1}{n}[x_i(fa)_k - x_k(fa)_i]_{(d)}[fa]$. From (42).

(90) $f^{(ik)}(fa)_\lambda(xa) = f_{ik}(fa)_\lambda(xa)$.

(91) $(fa)^{(ik)}f_l(xa) = (fa)_{ik}f_i(xa)$.

(92) $f_i(fa)_l(fa)_k = \frac{1}{n}x_i(fa)_k(fa)_l - f^{(ik)}(fa)_l(xa)$. From (55).

(93) $f_i(fa)_l(xa)_k = -[f^{(ik)}(fa)_l + f_i(fa)^{(kl)}](xa)$. From (33).

(94) $f_l(fa)^{(ik)}(xa) = -\frac{1}{n}x_l(fa)_i(fa)_k$. From (92) and (93).

(95) $f_i[(fa)_l(xa)_k - (fa)_k(xa)_l] = [f^{(il)}(fa)_k - f^{(ik)}(fa)_l](xa)$. From (92).

(96) $f_k((fa),(xa),u) = (xa)_k((fa),f,u) - \frac{1}{n}(fa)_k((fa),x,u)$. From (92).

(97) $f_k((fa),(xa),u) = (xa)_k((fa),f,u) - (fa)_k((xa),f,u)$. From (96).

(98) $f_k((fa),(xa),u) = (xa)(f(fa),u)_k + (fa)_k(f,(xa),u)$.

(99) $(f\phi u)((fa),(\phi a),v) = (f\phi v)((fa),(\phi a),u)$. From (97).

(100) $n(n-1)[(f_i\phi_k - f_k\phi_i)^{(l)}(xya) + (f_i\phi_k - f_k\phi_i)(xya)_l][f\phi a]$
$= 2[(x_iy_k - x_ky_i)^{(l)}(f\phi a) + (x_iy_k - x_ky_i)(f\phi a)_l][f\phi a]$. From (36).

(101) $\eta(f_i\phi_k - f_k\phi_i)(f\phi a)(xya)_\lambda = 2(x_iy_k - x_ky_i)^{(\lambda)}$. From (100).

(102) $n(n-1)[(f_i\phi_k - f_k\phi_i)^{(\lambda)}(xya) + (f_i\phi_k - f_k\phi_i)(xya)_\lambda](f\phi a)_\mu$
$= 2(x_iy_k - x_ky_i)(f\phi a)_\lambda(f\phi a)_\mu$. From (100).

(103) $[(f_i\phi_k - f_k\phi_i)^{(\lambda)}(xya) + (f_i\phi_k - f_k\phi_i)(xya)_\lambda]$
$= -(xya)(f_i\phi_k - f_k\phi_i)^{(\lambda)}(f\phi a)_\mu$. From (38).

(104) $n(n-1)(f_i\phi_k - f_k\phi_i)^{(\lambda)}(f\phi a)_\mu(xya)$
$= -2(x_iy_k - x_ky_i)(f\phi a)_\lambda(f\phi a)_\mu$. From (102) and (103).

The following formulas involving third covariantive derivatives are of special importance for the next article.

(105) $x^{(irs)} = \epsilon[\phi_i(\phi a)_s(xa)_r + \phi_i(\phi a)(xa)^{(rs)}]$. From (76).

(106) $x^{(irs)} - x^{(isr)} = \epsilon\phi_i[(\phi a)_s(xa)_r - (\phi a)_r(xa)_s]$.

We notice that in third covariantive derivatives the order of differentiation is not arbitrary.

(107) $x^{(irs)} - x^{(isr)} = \epsilon\left[\phi^{(is)}(\phi a)_r - \phi^{(ir)}(\phi a)_s\right](xa)$. From (106) and (92).

(108) $x^{(irs)} - x^{(isr)} = \frac{1}{4}\epsilon(\phi^{(irs)} - \phi^{(isr)})(\overset{1}{4}\phi a)(xa)$. From (107) and (84).

(109) $x^{(irs)} - x^{(isr)} = \epsilon^2\left[(\phi a)_r(\phi b)_s - (\phi a)_s(\phi b)_r\right]\psi_i(\psi b)(xa)$. From (107).

(110) $x^{(irs)} - x^{(isr)} = \epsilon^2\left[\psi_s(\phi a)_r - \psi_r(\phi a)_s\right](\psi b)(\phi b)_i(xa)$. From (107).

(111) $\left[x^{(irs)} - x^{(isr)}\right](fa)\left[fa\right] = n(f^{(irs)} - f^{(isr)})(xa)\left[fa\right]$.

(112) $(f^{(irs)} - f^{(isr)})(xa)(fa)_\lambda = 0$. From (111).

(113) $f_k(f^{(isr)} - f^{(irs)}) = f^{(ir)}f^{(sk)} - f^{(is)}f^{(rk)}$. From (84).

(114) $f^{(k\lambda)}(f^{(irs)} - f^{(isr)}) = 0$. From (76) and (112).

§ 8. *The quadrilinear covariant.*

In § 6 we found a quadratic covariant expression. It can be shown that the lowest covariant proper, i. e. one which does not contain any arbitrary function u, is quadrilinear and of the second order. This covariant occurs in RIEMANN's paper: *Commentatio mathematica*, etc., whose second part is devoted practically to the analytic deduction of the propositions established in the famous paper: *Ueber die Hypothesen welche der Geometrie zu Grunde liegen.* The quadrilinear covariant constitutes the numerator of a fraction which RIEMANN defines as the general measure of curvature.*

CHRISTOFFEL arrived quite independently † at the same covariant; it forms the basis of his deductions of the conditions for the equivalence of two quadratic differential quantics.

The covariant is defined in the four sets of differentials, $d^{(1)}x$, $d^{(2)}x$, $d^{(3)}x$, $d^{(4)}x$ as

(115) $$G_4 = \sum_{ikrs}(ikrs)\,d^{(1)}x_i\,d^{(2)}x_k\,d^{(3)}x_r\,d^{(4)}x_s,$$ ‡

where $(ikrs)$ is the *quadruple index symbol:*

(116) $$(ikrs) = \frac{\partial\begin{bmatrix} ir \\ k \end{bmatrix}}{\partial x_s} - \frac{\partial\begin{bmatrix} is \\ k \end{bmatrix}}{\partial x_r} + \beta^2\sum_{lm}A_{lm}\left\{\begin{bmatrix} is \\ m \end{bmatrix}\begin{bmatrix} rk \\ l \end{bmatrix} - \begin{bmatrix} ir \\ m \end{bmatrix}\begin{bmatrix} sk \\ l \end{bmatrix}\right\}.$$ §

* *Riemann's gesammelte Werke*, herausgegeben von H. WEBER, 2d ed. (Leipzig, 1892), pp. 403, 412.

† In 1869. RIEMANN's paper *Commentatio mathematica* was written in 1861, but not published until 1876, ten years after his death, by WEBER-DEDEKIND.

‡ CHRISTOFFEL, loc. cit., p. 58; BIANCHI, loc. cit., p. 50.

§ RIEMANN, loc. cit., pp. 402, 411; CHRISTOFFEL, loc. cit., p. 54; BIANCHI, loc. cit., p. 51.

If now we introduce symbols, we have

$$(ikrs) = f_{ir}f_{ks} - f_{is}f_{kr} + \frac{\beta^2}{(n-1)!}\sum_{lm} F^{1,l}\cdot F^{1,m}\cdot[f_{m}f_{is}\phi_l\phi_{rk} - f_m f_{ir}\phi_l\phi_{sk}],$$

which expression, considering that

$$\beta\sum\phi_i F^{1,l} = (\phi f) \text{ and } \beta\sum_m f_m F^{1,m} = (f),$$

is transformed into

$$\begin{aligned}(ikrs) &= f_{ir}f_{ks} - f_{is}f_{kr} + \epsilon(f_{ik}\phi_{kr} - f_{ir}\phi_{ks})(\phi a)(fa)\\ &= f_{ir}[f_{ks} - \epsilon\phi_{ks}(\phi a)(fa)] - f_{is}[f_{kr} - \epsilon\phi_{kr}(\phi a)(fa)]\\ &= f_{ir}f^{(ks)} - f_{is}f^{(kr)}.\end{aligned}$$

This on account of (76) and (85) finally leads to

(117) $$(ikrs) = f^{(ir)}f^{(ks)} - f^{(is)}f^{(kr)},$$

or also from (113),

(118) $$(ikrs) = f_k(f^{(isr)} - f^{(irs)}).$$

This is the simple symbolic representation of the quadruple index symbol $(ikrs)$. By means of the formulas (105)–(110) we deduce from it the following expressions:

(119) $$(ikrs) = \epsilon f_i\phi_k[(fa)_r(\phi a)_s - (fa)_s(\phi a)_r].$$

(120) $$(ikrs) = \epsilon(fa)_r(\phi a)_s(f_i\phi_k - f_k\phi_i).$$

(121) $$2(ikrs) = \epsilon(f_i\phi_k - f_k\phi_i)[(fa)_r(\phi a)_s - (fa)_s(\phi a)_r].$$

(122) $$(ikrs) = \epsilon^2[(\phi b)_r(\phi a)_s - (\phi a)_r(\phi b)_s](\psi b)(fa)\psi_i f_k.$$

(123) $$2(ikrs) = \epsilon^2(fa)(\psi b)(\psi_i f_k - \psi_k f_i)[(\phi b)_r(\phi a)_s - (\phi a)_r(\phi b)_s].$$

(124) $$(ikrs) = \epsilon^2(\psi b)(fa)[\psi_r(\phi a)_s - \psi_s(\phi a)_r]f_k(\phi b)_i.$$

(125) $$2(ikrs) = \epsilon^2(\psi b)(fa)[\psi_r(\phi a)_s - \psi_s(\phi a)_r][f_k(\phi b)_i - f_i(\phi b)_k].$$

To each of the parentheses appearing on the right sides of these expressions (119)–(125) we now apply formula (37), and thus we find seven expressions for the coefficients $(ikrs)$ of G_4, which lead at once to as many symbolic representations of G_4, furnishing at the same time the proof that G_4 is a covariant of A.

It will be sufficient to write one of these seven formulas down. From (119) we deduce

(126) $$(ikrs) = \epsilon\eta\big((fa)(\phi a)b\big)(f'\phi' b)f_i\phi_k f'_r\phi'_s,$$

and

(127) $$(n-1)!(n-2)!\,G_4 = \big((fa)(\phi a)b\big)(f'\phi' b)f_{x_1}\phi_{x_2}f'_{x_3}\phi'_{x_4}.$$

§9. *Higher covariants.*

For the general theory and for CHRISTOFFEL's algebraic theory in particular there are of paramount importance besides G_4 certain covariants G_5, G_6, $\cdots$ which are linear in 5, 6, $\cdots$ sets of differentials. The process by which CHRISTOFFEL deduces the coefficients of $G_{\mu+1}$ from those of G_μ appears rather complicated. We shall see, however, that *this process consists simply in one single covariantive differentiation.*

CHRISTOFFEL denotes the coefficients of G_μ by

$$(i_1 i_2 \cdots i_\mu)$$

and defines the coefficient of $G_{\mu+1}$ by

$$(ii_1 i_2 \cdots i_\mu) = \frac{\partial (i_1 i_2 \cdots i_\mu)}{\partial x_i} - \sum_\lambda \left[\left\{ \begin{matrix} i_1 i \\ \lambda \end{matrix} \right\} (\lambda i_2 \cdots i_\mu) + \left\{ \begin{matrix} i_2 i \\ \lambda \end{matrix} \right\} (i_1 \lambda \cdots i_\mu) + \cdots \right]^* . \tag{128}$$

Let us suppose now that $(i_1 i_2 \cdots i_\mu)$ is representable as a sum of terms of the form

$$T = M u^1_{i_1} u^2_{i_2} \cdots u^\mu_{i_\mu}, \tag{129}$$

where M denotes any quantity for which the law

$$M^{(\lambda)} = M_\lambda$$

holds and where

$$u^1_{i_1} = \frac{\partial u^1}{\partial x_{i_1}} \text{ etc.}$$

Applying now to every T formula (128) we obtain

$$T_i - M \left[u^1_{i_2} \cdots u^\mu_{i_\mu} \sum_\lambda u^1_\lambda \left\{ \begin{matrix} i_1 i \\ \lambda \end{matrix} \right\} + u^1_{i_1} u^3_{i_3} \cdots u^\mu_{i_\mu} \sum u^2_\lambda \left\{ \begin{matrix} i i_2 \\ \lambda \end{matrix} \right\} + \cdots \right],$$

which reduces by means of (76^a) to

$$T_i - M \left[u^1_{i_1 i} u^2_{i_2} \cdots u^\mu_{i_\mu} + u^1_{i_1} u^2_{i_2 i} \cdots u^\mu_{i_\mu} + \cdots - (u^{1(i_1 i)} u^2_{i_2} \cdots u^\mu_{i_\mu} + \cdots) \right],$$

i. e., to $T^{(i)}$ since $M^{(i)} = M_i$.

We have then

$$(ii_1 i_2 \cdots i_\mu) = (i_1 i_2 \cdots i_\mu)^{(i)}. \tag{130}$$

When we apply to the second covariantive derivatives of the u's formula (76) it becomes evident that $T^{(i)}$ consists also of terms of the type (129). The same

* Loc. cit., p. 57.

is therefore true for $(ii_1 i_2 \cdots i_\mu)$. Finally, as (126) shows, $(i_1 i_2 i_3 i_4)$, i. e., our $(ikrs)$ is directly given considering equation (83) in the form (129). Consequently equation (130) is proved for every $\mu > 3$.

For our purpose it is necessary to represent G_μ as a single product of invariantive and covariantive constituents — like G_4 in (127) — i. e., we have to represent the coefficients of G_μ as single products of the form (129) where the factor M consists of a product of invariantive constituents.*

For $\mu = 5$ this representation can be effected as follows: From (130), (118) and (114) we deduce at once for coefficients of G_5 the simple symbolic form

$$(\lambda ikrs) = f_k (f^{(irs\lambda)} - f^{(irs\lambda)}).$$

We obtain by differentiating (119) covariantively

$$(\lambda ikrs) = \epsilon f_i \phi_k [(fa)_r (\phi a)_s - (fa)_s (\phi a)_r]^{(\lambda)} + \epsilon (f_i \phi_k)^{(\lambda)} [(fa)_r (\phi a)_s + (fa)_s (\phi a)_r].$$

The second term on the right side vanishes because, for instance,

$$f^{(i\lambda)} (fa)_r = -f^{(i\lambda r)} (fa) \text{ from (84)},$$

and

$$\phi_k (\phi a)_s (fa) = 0 \text{ from (33)}.$$

From (101) we find

$$[(fa)_r (\phi a)_s - (fa)_s (\phi a)_r]^{(\lambda)} = \eta f'_r \phi'_s (f' \phi' b) \big((fa)(\phi a) b\big)_\lambda,$$

and from (32)

$$\big((fa)(\phi a) b\big)_\lambda = 2 f''_\lambda (f'' c) \Big(\big((fa)(\phi a) b\big) c\Big),$$

hence

$$(131) \qquad (\lambda ikrs) = \epsilon^2 \eta \Big(\big((fa)(\phi a) b\big) c\Big) (f' \phi' b) (f'' c) f_i \phi_k f'_r \phi'_s f''_\lambda,$$

and

$$(132) \quad (n-1)!\,(n-1)!\,(n-2)!\; G_5 = \Big(\big((fa)(\phi a) b\big) c\Big) (f' \phi' b) (f'' c) f_{x^1} \phi_{x^2} f'_{x^3} \phi'_{x^4} f''_{x^5}.$$

§ 10. *The three invariants of the second order for $n = 3$.*

In the case $n = 2$ there exists only one invariant (proper) of the second order, viz., the Gaussian curvature; for $n = 3$ there are three invariants,† viz.,

* I have not yet been able to overcome this difficulty in a satisfactory manner for $\mu > 5$.

† Compare HASKINS, *On the invariants of quadratic differential forms*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 86, footnote.

$$H_1 = \frac{|B_{11}A_{22}A_{33}| + |A_{11}B_{22}A_{33}| + |A_{11}A_{22}B_{33}|}{|A_{11}A_{22}A_{33}|},$$

$$H_2 = \frac{|A_{11}B_{22}B_{33}| + |B_{11}A_{22}B_{33}| + |B_{11}B_{22}A_{33}|}{|A_{11}A_{22}A_{33}|},$$

$$H_3 = \frac{|B_{11}B_{22}B_{33}|}{|A_{11}A_{22}A_{33}|},$$

where the A_{ik} are the quantities (18) and the B_{ik} the quadruple index symbols, viz.:

$$B_{11} = (2323), \quad B_{12} = (2331), \quad B_{13} = (2312),$$
$$B_{22} = (3131), \quad B_{23} = (3112), \quad B_{33} = (1212).$$

We want to find the symbolic representation of H_1, H_2, H_3.

From

$$|A_{11}A_{22}A_{33}| = \beta^{-4} \qquad \text{and} \qquad \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = \beta^{-2} a_{11}$$

we obtain for the first term of H_1

$$\beta^2 \sum_k a_{1k} B_{1k},$$

which by means of (120) becomes equal to

$$\tfrac{1}{2}\beta^2 (fa)_2 (\phi a)_3 \begin{vmatrix} f_1 & f_2 & f_3 \\ \phi_1 & \phi_2 & \phi_3 \\ a_{11} & a_{12} & a_{13} \end{vmatrix};$$

and by replacing the a_{ik} by the symbolic products $b_i b_k$, equal to

$$\tfrac{1}{2}\beta b_1 (fa)_2 (\phi a)_3 (f\phi b).$$

Thus we have

$$2H_1 = \beta [b_1 (fa)_2 (\phi a)_3 + b_2 (fa)_3 (\phi a)_1 + b_3 (fa)_1 (\phi a)_2] (f\phi\psi),$$

which leads at once to the required representation

$$4H_1 = \big((fa)(\phi a) b\big)(f\phi b). \tag{133}$$

For the computation of H_2 and H_3 we need the following theorem on determinants. The notations P_{ik}, etc., denoting the minors of the elements p_{ik}, etc., in the determinants

$$|p_{ik}|, \qquad |q_{ik}|, \qquad |r_{ik}|$$

the relation holds

$$\begin{vmatrix} P_{11} & P_{12} & P_{13} \\ Q_{11} & Q_{12} & Q_{13} \\ R_{11} & R_{12} & R_{13} \end{vmatrix} = \begin{vmatrix} p_{21} & p_{22} & p_{23} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} \cdot \begin{vmatrix} p_{31} & p_{32} & p_{33} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} - \begin{vmatrix} p_{21} & p_{22} & p_{23} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} \cdot \begin{vmatrix} p_{31} & p_{32} & p_{33} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix}. \tag{134}$$

In $$H_2 = \beta^4 \{ |A_{11} B_{22} B_{33}| + |B_{11} A_{22} B_{33}| + |B_{11} B_{22} A_{33}| \}$$

we now replace A_{11} by $f_2\phi_3(f_2\phi_3 - f_3\phi_2)$, etc.,

and the B_{ik} by the quantities given above where, however, we have here to use two different sets of symbols: f', ϕ', a' and f'', ϕ'', a''.

Using the abbreviation

$$(fa)' = (f'a'), \text{ etc.},$$

we find

$$4|A_{11}B_{22}B_{33}| = \beta^4(f_2\phi_3 - f_3\phi_2)[(fa)'_3(\phi a)'_1 - (fa)'_1(\phi a)'_3]$$
$$\times [(fa)''_1(\phi a)''_2 - (fa)''_2(\phi a)''_1] \cdot \begin{vmatrix} f_2\phi_3, & f_3\phi_1, & f_1\phi_2 \\ f'_2\phi'_3, & f'_3\phi'_1, & f'_1\phi'_2 \\ f''_2\phi''_3, & f''_3\phi''_1, & f''_1\phi''_2 \end{vmatrix},$$

and from this

$$8H_2 = \beta^4 \begin{vmatrix} f_2\phi_3 - f_3\phi_2, & f_3\phi_1 - f_1\phi_3, f_1\phi_2 - f_2\phi_1 \\ (fa)'_2(\phi a)'_3 - (fa)'_3(\phi a)'_2, & \cdots, \quad \cdots \\ (fa)''_2(\phi a)''_3 - (fa)''_3(\phi a)''_2, & \cdots, \quad \cdots \end{vmatrix}$$
$$\times \begin{vmatrix} f_2\phi_3, & f_3\phi_1, & f_1\phi_2 \\ f'_2\phi'_3, & f'_3\phi'_1, & f'_1\phi'_2 \\ f''_2\phi''_3, & f''_3\phi''_1, & f''_1\phi''_2 \end{vmatrix}$$

which can be transformed by proper permutation of equivalent symbols into another expression where the elements $f_2\phi_3$, etc., of the second determinant are replaced by $f_2\phi_3 - f_3\phi_2$, etc. Applying now (134) we have

$$64H_2 = \big[\big((fa)'(\phi a)'f\big)\big((fa)''(\phi a)''\phi\big)$$
$$- \big((fa)'(\phi a)'\phi\big)\big((fa)''(\phi a)''f\big)\big][(f'\phi'f)(f''\phi''\phi)$$
$$- (f'\phi'\phi)(f''\phi''f)],$$

and permuting f and ϕ in the last term

$$32H_2 = \big[\big((fa)'(\phi a)'f\big)\big((fa)''(\phi a)''\phi\big)$$
$$- \big((fa)'(\phi a)'\phi\big)\big((fa)''(\phi a)''f\big)\big](f'\phi'f)(f''\phi''\phi). \tag{135}$$

By using (62) for the transformation of the bracket we obtain the final form:

$$32H_2 = \big((fa)'(\phi a)'b'\big)\big((fa)''(\phi a)''b''\big)(bb'b'')(bf\phi)(ff'\phi')(\phi f''\phi''). \tag{136}$$

To compute the third invariant

$$H_3 = \beta^4 |B_{11}B_{22}B_{33}|$$

we take formula (126) for the B_{ik}. On setting, temporarily

$$(\psi\chi b)(\psi'\chi' b')(\psi''\chi'' b'') = (\psi\chi b)_{012}$$

and using the notation

$$\big((fa)(\phi a)b\big)_{012}$$

in a similar sense, we find

$$8H_3 = \beta^4\big((fa)(\phi a)b\big)_{012}\cdot(\psi\chi b)_{012}$$

$$(137)\qquad \cdot\psi_2\chi_3'\psi_3'\chi_1'\psi_1''\chi_2'' \begin{vmatrix} f_2\phi_3 - f_3\phi_2, & f_3\phi_1 - f_1\phi_3, & f_1\phi_2 - f_2\phi_1 \\ f_2'\phi_3' - f_3'\phi_2', & \cdots, & \cdots, \\ f_2''\phi_3'' - f_3''\phi_2'', & \cdots, & \cdots, \end{vmatrix}.$$

In this expression we can replace

$$(\psi\chi b)_{012}\cdot\psi_2\chi_3\psi_3'\chi_1'\psi_1''\chi_2''$$

by

$$(138)\qquad \tfrac{1}{6}(\psi\chi b)_{012}\begin{vmatrix} \psi_2\chi_3, & \psi_3\chi_1, & \psi_1\chi_2 \\ \psi_2'\chi_3', & \psi_3'\chi_1', & \psi_1'\chi_2' \\ \psi_2''\chi_3'', & \psi_3''\chi_1'', & \psi_1''\chi_2'' \end{vmatrix}.$$

But

$$\psi_2\chi_3(\psi\chi b) = \beta(b_1A_{11} + b_2A_{12} + b_3A_{13}),$$

hence (138) becomes equal to

$$\tfrac{1}{6}\beta^3|A_{ik}|\cdot\frac{1}{\beta}(bb'b'') = \frac{1}{6\beta^2}(bb'b'').$$

We find now by applying theorem (134) to the determinant in (137)

$$48H_3 = \big((fa)(\phi a)b\big)_{012}(bb'b'')\,[(ff'\phi')(\phi f''\phi'') - (ff''\phi'')(\phi f'\phi')],$$

and finally by permuting f and ϕ in the last term

$$(139)\quad 24H_3 = \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)\big((fa)''(\phi a)''b''\big)(ff'\phi')(\phi f''\phi'')(bb'b'').$$

In (133), (136) and (139) we have now the symbolic representation of the three ternary invariants H_1, H_2, H_3.

§ 11. *The three simplest general invariants of the second order.*

The formula for the ternary invariant H_1 can at once be generalized for n variables by letting the letter b stand for the $n-2$ symbols $b^3, b^4, \cdots b^n$. I denote this invariant by R_1, for a reason which will appear presently.

$$(140)\qquad R_1 = \big((fa)(\phi a)b\big)(f\phi b).$$

We have then

$$H_1 = 4R_1 \qquad \text{for} \qquad n = 3,$$

and $2K = R_1$ for $n = 2\,(b = 0)$, where K denotes the Gaussian curvature. To transform H_2 first, we start from (135) and multiply through by $(f'\phi' f)(f''\phi''\phi)$. The first term is then simply equal to R_1^2 and with a slight change of notation in the second term we have

$$32H_2 = R_1^2 - \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)(f\phi b')(f'\phi' b).$$

The invariant representing the second term can now be generalized at once by letting the letters b' and b'' run out. We denote it by R_2.

$$R_2 = \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)(f\phi b')(f'\phi' b). \tag{141}$$

We have then

$$32H_2 = R_1^2 - R_2 \qquad \text{for} \qquad n = 3.$$

For the transformation of H_3 we deduce from the identity (79) the equation

$$(bb'b'')(ff'\phi') = (bb'f)(b''f'\phi') + (b'b''f)(bf'\phi') + (b''bf)(b'f'\phi'), \tag{142}$$

and also

$$(bb'f)(\phi f''\phi'') = (b'f\phi)(bf''\phi'') - (bf\phi)(b'f''\phi'') + (bb'\phi)(ff''\phi''). \tag{143}$$

Multiplying now (142) by $(\phi f''\phi'')$ and applying formula (143) to the first term of the product, we can, after substitution into (139), permute f and ϕ in the term arising from the last term of (143). By this permutation the last term of (143) becomes equal to the negative term on the left side, so that in the complete expression of H_3 we can use instead of (143) the equation

$$2(bb'f)(\phi f''\phi'') = (b'f\phi)(bf''\phi'') - (bf\phi)(b'f''\phi''),$$

and two others obtained from it by cyclical permutations of b, b', b''. Combining, we can now in H_3 use the equation

$$\begin{aligned} 2(bb'b'')(ff'\phi')(\phi f''\phi'') &= (bf''\phi'')(b'f\phi)(b''f'\phi') \\ &\quad + (bf'\phi')(b'f''\phi'')(b''f\phi) + (bf\phi)(b'f'\phi')(b''f''\phi'') \\ &\quad - [(bf\phi)(b'f''\phi'')(b''f'\phi') + (bf'\phi')(b'f\phi)(b''f''\phi'') \\ &\quad + (bf''\phi'')(b'f'\phi')(b''f\phi)]. \end{aligned}$$

This leads to

$$\begin{aligned} 48H_3 = \big((fa)(\phi a)b\big)_{012}[&(f\phi b')(f'\phi' b'')(f''\phi'' b) \\ &+ (f\phi b'')(f'\phi' b)(f''\phi'' b')] + R_1^3 - 3R_1R_2. \end{aligned}$$

The second term becomes equal to the first by the interchange of the quantities with one and two accents. This gives rise to the introduction of the following invariant R_3 which now also holds for every n if we again let the letters b, b', b'', run out.

$$(144)\quad R_3=\big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)\big((fa)''(\phi a)''b''\big)(f\phi b')(f'\phi' b'')(f''\phi'' b).$$

We have now

$$48H_3=2R_3+R_1^3-3R_1R_2 \qquad \text{for} \qquad n=3.$$

In R_1, R_2, R_3, *given by* (140), (141) *and* (144) *we have then the three simplest invariants of* n *variables.*

THE UNIVERSITY OF CHICAGO,
June, 1903.

CONGRUENCES OF CURVES*

BY

LUTHER PFAHLER EISENHART

In the second volume of his *Leçons* DARBOUX has considered congruences of curves defined by the equations,

$$f(x, y, z, a, b) = 0, \qquad \phi(x, y, z, a, b) = 0,$$

where x, y, z are the rectangular coördinates of a point on one of these curves and a, b are the parameters of the curves, and he has established many interesting theorems about congruences of this kind.

In the present discussion we shall consider such congruences when defined by equations of the form,

$$(1) \qquad x = f_1(t, u, v), \qquad y = f_2(t, u, v), \qquad z = f_3(t, u, v),$$

where u and v denote the parameters determining the curves and t is the parameter which determines the points on the curve. By this method of definition the equations of a congruence of curves are given a form similar to that usually adopted in the discussions of rectilinear congruences, so that one proceeds naturally to a generalization of some of the properties of the latter. At the same time the curves, as above defined, are looked upon as the intersections of surfaces, namely, those defined by (1), when u and v respectively are constants.

In §§ 1, 2 we show how the theorems of DARBOUX can be established when the congruence is defined in this manner. Several examples are given to illustrate the different theorems.

In § 3 we consider the problem of determining a function $\phi(u, v)$ such that the tangents to the curves of the congruence at the points of intersection with the surface, defined by (1) after t has been replaced by ϕ, shall form a normal congruence, or in the second place the ruled surfaces $u = \text{const.}$, $v = \text{const.}$ of this congruence of tangents shall be developable. It is shown that for every congruence a function exists which furnishes a solution to the first of these problems, and the condition is found which the functions f_1, f_2, f_3 must satisfy in order that ϕ may be constant. However, there does not always exist a function such that the second condition is satisfied. But when the curves $t = \text{const.}$,

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$u = \text{const.}$ on the surfaces defined by (1) when v is constant form a conjugate system, and when the curves $t = \text{const.}$, $v = \text{const.}$ on the surfaces defined by (1) when u is constant are likewise conjugate, ϕ has a constant value and only in this case.

It is next proposed to find a function $\phi(u)$ such that all the tangents to the curves whose parameters satisfy the relation $v = \phi(u)$ form a normal congruence. It is shown that such a function cannot be found for every congruence, but exists only in the case of normal rectilinear congruences when equations (1) define space referred to a triply orthogonal system of surfaces. In closing § 4 we remark that the tangents to all the curves of a congruence form a complex and we find the condition that this complex be linear.

The formulæ (1) define space referred to a triple system of surfaces $t = \text{const.}$, $u = \text{const.}$, $v = \text{const.}$ and in the preceding sections we have considered (1) as defining the congruence C_t of curves of intersection of the surfaces $u = \text{const.}$, $v = \text{const.}$ It is evident, however, that these equations define equally well the congruence C_u and C_v composed of the intersections of the surfaces $t = \text{const.}$, $v = \text{const.}$, and $t = \text{const.}$, $u = \text{const.}$, respectively. Therefore we say that equations (1) define a *triple congruence* of curves. In § 5 the different theorems and equations of the preceding sections are viewed in this three-fold light and several interesting theorems are established, notably that the focal surface of all three congruences is the same surface. The most general triply rectilinear congruence is considered and a few of its properties noted.

In § 6 we consider the triple congruences such that each congruence is cut orthogonally by a family of surfaces, and point out the relation of this problem to that of solving the equations in the study of applicable surfaces.

Finally, in § 7, the conditions are determined which the functions f_1, f_2, f_3 must satisfy in order that the three complexes of tangents be linear.

§ 1. Surfaces of the Congruence. Focal Points and Focal Surfaces.

Consider a congruence of curves defined by equations of the form

$$(1) \qquad x = f_1(t, u, v), \qquad y = f_2(t, u, v), \qquad z = f_3(t, u, v),$$

where the parameters u and v determine the curve and t a point upon the curve. We shall assume that the functions f_1, f_2, f_3 and their derivatives of the first and second orders with respect to these variables are finite and continuous within a suitable region.

It is clear that, when f_1, f_2, f_3 are algebraic, there pass through a given point in space a number, in general limited, of curves of the congruence. For, when particular values are given to x, y, z, the values of t, u, v given by equa-

tions (1) will in general be finite in number. However, when this restriction is not put upon these functions, it will not be unusual to have an infinity of the curves through a point. As an example of this, as DARBOUX remarks,* the totality of tangents to a surface from points on a given curve is a rectilinear congruence having an infinity of lines through each point of the curve.

If we establish a relation between u and v, say

$$v = \phi(u) \tag{2}$$

the curves of the congruence whose parameters satisfy this relation depend upon the single parameter u and consequently form a surface; the coördinates of this surface are given by (1), when v has been replaced by ϕ. This surface will evidently be changed as ϕ is given different forms. With DARBOUX we shall call these *surfaces of the congruence*. It is evident that in the case of rectilinear congruences all these surfaces are ruled.

The equation of the tangent plane to the surface defined by (1) and (2) is

$$\begin{vmatrix} \xi - x & \eta - y & \zeta - z \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \\ \dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v}\phi'(u) & \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}\phi'(u) & \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\phi'(u) \end{vmatrix} = 0, \tag{3}$$

where ξ, η, ζ are current coördinates and the accent denotes differentiation. This equation can be written in the form

$$\begin{aligned} 0 = {} & (\xi - x)\left[\frac{\partial(y,z)}{\partial(t,u)} + \phi'(u)\frac{\partial(y,z)}{\partial(t,v)}\right] \\ & + (\eta - y)\left[\frac{\partial(z,x)}{\partial(t,u)} + \phi'(u)\frac{\partial(z,x)}{\partial(t,v)}\right] + (\zeta - z)\left[\frac{\partial(x,y)}{\partial(t,u)} + \phi'(u)\frac{\partial(x,y)}{\partial(t,v)}\right]. \end{aligned} \tag{4}$$

It is readily found that the anharmonic ratio of the tangent planes to four surfaces of the congruence through a given curve and corresponding to functions $\phi_1, \phi_2, \phi_3, \phi_4$ depends only upon these functions, so that we have the theorem of DARBOUX:†

For any four surfaces of a congruence containing the same curve of the congruence the anharmonic ratio of the tangent planes to these surfaces at a point is constant when the point moves along the curve.

From (4) it is seen that when x, y, z satisfy the condition

* *Leçons*, vol. 2, p. 2.

† *Leçons*, vol. 2, p. 3.

$$\frac{\partial(y,z)}{\partial(t,u)}=\frac{\partial(z,x)}{\partial(t,u)}=\frac{\partial(x,y)}{\partial(t,u)}=\frac{\partial(y,z)}{\partial(t,v)}=\frac{\partial(z,x)}{\partial(t,v)}=\frac{\partial(x,y)}{\partial(t,v)}=\lambda, \tag{5}$$

the equation of the tangent plane is independent of the function ϕ and consequently is the same for all surfaces of the congruence through the corresponding points. This gives the theorem: *

Upon a curve of a congruence there are points at which all the surfaces of the congruence through the curve admit the same tangent plane, whatever be the law according to which the curves have been assembled to generate the surfaces.

With DARBOUX we shall refer to these points as the *focal points* and to their locus as the *focal surface.* In general this focal surface will consist of sheets, whose number will depend upon the number of focal points on the curves.

The equation (5) may be replaced by

$$\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0, \tag{6}$$

from which we have the relation

$$\frac{\partial x}{\partial t}=l\frac{\partial x}{\partial u}+m\frac{\partial x}{\partial v}, \tag{7}$$

and similar ones in y and z.

The equation (6) can be looked upon as an equation in t having its coefficients functions of u and v. When u and v are given particular values, the values of t satisfying this equation define the focal points upon the corresponding line. When equation (6) is solved for t and the various solutions are substituted in (1) the latter will define the sheets of the focal surface. For example, suppose that the congruence is rectilinear, in which case (1) can be given the form

$$x=a_1+a_2t, \qquad y=b_1+b_2t, \qquad z=c_1+c_2t, \tag{8}$$

where $a_1, \cdots, c_2$ are functions of u and v. The corresponding equation (6) is a quadratic in t. When the two roots of this equation are substituted in (8), we have the equations of the two focal sheets.

* DARBOUX, l. c., p. 4.

§ 2. Singular Lines and Singular Surfaces. Principal Surfaces.

It is evident that the character of the functions f_1, f_2, f_3 will determine whether the number of the focal points on a curve is finite or infinite. Again, it may happen that for particular values of u and v, that is for particular lines, the equation (6) will be satisfied identically so that t is indeterminate. In this case every point of the curve is a focal point. When such lines exist, they will in general be isolated; we shall call them *singular lines* of the congruence. In some cases, however, there is a continuum of these lines and consequently a *singular surface.*

Consider for example the congruence defined by

$$\begin{aligned} x &= A(a-t)^m(a-u)^n(a-v)^p, \\ y &= B(b-t)^m(b-u)^n(b-v)^p, \\ z &= C(c-t)^m(c-u)^n(c-v)^p, \end{aligned} \tag{9}$$

where A, B, C, a, b, c are constants. For u or v constant these formulæ define a tetrahedral surface, so that the curves of the congruence are the intersections of two families of tetrahedral surfaces. On this account we shall call the congruences defined by (9) *tetrahedral congruences.* Substituting the above values for x, y, z in equation (6), we have

$$\begin{vmatrix} (a-u)(a-v) & (b-u)(b-v) & (c-u)(c-v) \\ (a-t)(a-v) & (b-t)(b-v) & (c-t)(c-v) \\ (a-t)(a-u) & (b-t)(b-u) & (c-t)(c-u) \end{vmatrix} = 0, \tag{10}$$

which reduces to

$$(t-u)(t-v)(u-v)(a-b)(a-c)(b-c) = 0. \tag{11}$$

From this we see that the focal points on the line $u = \alpha, v = \beta$, where α and β are two unequal constants, are given by $t = \alpha$ and $t = \beta$; but for all the curves whose parameters satisfy the condition $u = v$, the parameter t is indeterminate, and consequently these lines are singular lines. In this case we have a singular surface; its coördinates are

$$\begin{aligned} x &= A(a-t)^m(a-u)^{n+p}, \qquad y = B(b-t)^m(b-u)^{n+p}, \\ z &= C(c-t)^m(c-u)^{n+p}. \end{aligned} \tag{12}$$

Moreover, the coördinates of the focal surfaces proper have the expressions

$$\begin{aligned} x &= A(a-u)^{m+n}(a-v)^p, \qquad y = B(b-u)^{m+n}(b-v)^p, \\ z &= C(c-u)^{m+n}(c-v)^p; \end{aligned} \tag{13}$$

$$\begin{aligned} x &= A(a-u)^n(a-v)^{m+p}, \qquad y = B(b-u)^n(b-v)^{m+p}, \\ z &= C(c-u)^n(c-v)^{m+p}. \end{aligned} \tag{14}$$

From these three sets of expressions we see that the focal surfaces and the singular surface are tetrahedral surfaces. When the focal surface is defined as the locus of all points satisfying equation (6), it has all three of the above surfaces for nappes.

It should be remarked that every congruence defined by (9) does not have a singular surface, for there are certain values of the exponents which do not allow the equality of u and v. Such a case arises when $m = n = p = \frac{1}{2}$; then the congruence is composed of the curves of intersection of two families of confocal quadrics. If m, n, p be replaced by $\frac{1}{2}$ in (13) and (14), the latter take such a form that, if the parameters be eliminated, we have the same equation for the surface in each case and consequently there is only one equation to define the two nappes. This property is common to all congruences for which $m = n = p$.

Instead of solving equation (6) for t and substituting in (1) to get the equations of the focal surface, we can look upon equations (1), (7) and the two similar to the latter as defining the surface. A displacement upon this focal surface will have for projections on the coördinate axes the expression

$$dx = (ldt + du)\frac{\partial x}{\partial u} + (mdt + dv)\frac{\partial x}{\partial v},$$

and similar ones in y and z. Eliminating $ldt + du$ and $mdt + dv$ from these three equations, we get the equation of the tangent plane to the focal surface in a form which in consequence of equation (7) is reducible to the equation of the common tangent plane to the surfaces of the congruence at a focal point. Hence the theorem of DARBOUX: *

The surfaces of the congruence which contain a given curve admit the same tangent plane at a focal point as the focal surface and consequently the curves of the congruence are tangent at all their focal points to the focal surface.

In order that there exist surfaces of the congruence which are such that consecutive curves of the congruence upon these surfaces intersect and thus have an envelope, it is necessary that there be a relation of the form

$$dv = \phi(u, v)\,du, \tag{15}$$

such that †

$$\frac{\dfrac{\partial x}{\partial t}}{\dfrac{\partial x}{\partial u} + \phi\dfrac{\partial x}{\partial v}} = \frac{\dfrac{\partial y}{\partial t}}{\dfrac{\partial y}{\partial u} + \phi\dfrac{\partial y}{\partial v}} = \frac{\dfrac{\partial z}{\partial t}}{\dfrac{\partial z}{\partial u} + \phi\dfrac{\partial z}{\partial v}}. \tag{16}$$

* l. c. p. 5.

† PICARD, *Traité d'Analyse*, vol. 1, p. 294.

Solving for ϕ, we get

$$\phi = \frac{\partial(x,y)}{\partial(u,t)} = \frac{\partial(y,z)}{\partial(u,t)} = \frac{\partial(z,x)}{\partial(u,t)} \tag{17}$$
$$\frac{\partial(x,y)}{\partial(t,v)} \quad \frac{\partial(y,z)}{\partial(t,v)} \quad \frac{\partial(z,x)}{\partial(t,v)}$$

We have seen that the latter equalities are satisfied only at the focal points, and that for each focal point there is a solution $t = \psi(u, v)$ of equation (6), which is equivalent to the preceding equations. When such value for t is substituted in (17), we get a corresponding function $\phi(u, v)$ which furnishes a solution of our problem. Hence we have the following theorem similar to one given by DARBOUX:*

Corresponding to the focal points of a curve of a congruence there are surfaces of the congruence, through the given curve, whose consecutive curves have an envelope. And these envelopes lie on the sheets of the focal surface corresponding to the respective focal points.

We call these the *principal surfaces* of the congruence.

From this theorem and a preceding result concerning rectilinear congruences, it follows that every congruence of right lines has two principal surfaces and these are developable.

Let us suppose that the surface defined by (1) with $v = \text{const.}$ and passing through a given curve is a principal surface of the congruence. Then from (15) it follows that the corresponding function ϕ is zero. In this case the numerators in (17) are zero. But these expressions are proportional to the direction cosines of the normal to the surface $v = \text{const.}$ Hence along the curve on this surface which is the envelope of the curves of the congruence upon this surface the normal is indeterminate, so that this envelope is a singular line for the surface.

An example of this is furnished by the tetrahedral congruence. Thus from (9) we have

$$\frac{\partial(x,y)}{\partial(t,u)} = mnAB(a-t)^{m-1}(a-u)^{n-1}(a-v)^{p}(b-t)^{m-1}(b-u)^{n-1}(b-v)^{p}(u-t)(a-b),$$

and similarly for the other Jacobians. But from (11) we have that the foci are given by $t = u$ or $t = v$. The former makes the above zero, so that we have that the surfaces $v = \text{const.}$ are principal surfaces for the congruence and the curve $t = u$ on each of these surfaces is a singular line for the surface. It is to be remarked that in a similar manner the Jacobians $\partial(x, y)/\partial(t, v)$, etc., vanish for $t = v$, so that ϕ is infinite and hence the surfaces defined by (9) when u is constant are principal surfaces for the tetrahedral congruence, and the line $t = v$ is a singular line on each surface. It need hardly be mentioned that

* *Leçons*, vol. 2, p. 7.

these singular lines, and consequently the focal points, are not always real. This fact is evidenced in some of the tetrahedral congruences. Thus the central quadrics may be defined by

$$(18)\quad x^2=\frac{a(a-u)(a-t)}{(a-b)(a-c)},\quad y^2=\frac{b(b-u)(b-t)}{(b-a)(b-c)},\quad z^2=\frac{c(c-u)(c-t)}{(c-a)(c-b)},$$

where $a>b>c$. From these expressions it is seen that, if the tetrahedral surfaces $v=$ const. are central quadrics, the coördinate y of points on the line $t=u$ are imaginary.

§ 3. Certain Rectilinear Congruences Associated with Congruences of Curves.

When the formulæ (1) define space referred to a triply-orthogonal system of surfaces, the curves of the congruence under discussion are the intersections of the surfaces $u=$ const., $v=$ const. of the system. Since the surfaces are triply-orthogonal, these curves cut the surfaces $t=$ const. orthogonally and consequently the tangents to these curves at the points of intersection with one of the surfaces $t=$ const. from a normal congruence. Furthermore, the curves $u=$ const. and $v=$ const. on this surface are the lines of curvature and therefore the ruled surfaces $u=$ const., $v=$ const. of this congruence of tangents are developable. This suggests a two-fold problem of which the preceding case is a solution: *To determine, when possible, a function $\phi(u,v)$ such that if tangents be drawn to the curves of the congruence* (1) *at the points where they are met by the surface,*

$$(19)\qquad \xi=f_1(\phi,u,v),\qquad \eta=f_2(\phi,u,v),\qquad \zeta=f_3(\phi,u,v),$$

1° *these tangents form a normal congruence; or*

2° *the ruled surfaces $u=$ const., $v=$ const. are developable.*

The direction-cosines of the tangent to the curve whose coördinates have the expressions $f_1(u_0,v_0,t)$, $f_2(u_0,v_0,t)$, $f_3(u_0,v_0,t)$ are equal to

$$(20)\qquad \frac{\frac{\partial f_1}{\partial t},\ \frac{\partial f_2}{\partial t},\ \frac{\partial f_3}{\partial t}}{\sqrt{\left[\Sigma\left(\frac{\partial f_1}{\partial t}\right)^2\right]}}.$$

From the theory of rectilinear congruences we know that the necessary and sufficient condition that the congruence of tangents be normal to a surface is that *

$$\frac{\Sigma\frac{\partial f_1}{\partial t}\,d\xi}{\sqrt{\left[\Sigma\left(\frac{\partial f_1}{\partial t}\right)^2\right]}}+dl=0,$$

* Bianchi, *Lezioni*, p. 256; German edition, p. 268.

where l is the distance from the point (ξ, η, ζ) to the point where the line cuts an orthogonal surface. If we put

$$(21) \qquad \Sigma\left(\frac{\partial f_1}{\partial t}\right)^2 = T_1, \qquad \Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial u} = T_2, \qquad \Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial v} = T_3,$$

the above expression takes the form

$$(22) \qquad T_1 dt + T_2 du + T_3 dv + \sqrt{T_1}\, dl = 0.$$

When, in particular, the surface defined by (19) is orthogonal to the congruence the above equation reduces to

$$(23) \qquad T_1 dt + T_2 du + T_3 dv = 0.$$

Any function $t = \phi(u, v)$ which satisfies this equation gives a solution of our problem. In order that ϕ may involve an arbitrary constant, that is, in order that there may be an infinity of surfaces cutting the congruence in such a way that the tangents to the curves at the points of intersection are normal to the surfaces, it is necessary and sufficient that the functions T_1, T_2, T_3 satisfy identically the equation *

$$(24) \qquad T_1\left(\frac{\partial T_2}{\partial v} - \frac{\partial T_3}{\partial u}\right) + T_2\left(\frac{\partial T_3}{\partial t} - \frac{\partial T_1}{\partial v}\right) + T_3\left(\frac{\partial T_1}{\partial u} - \frac{\partial T_2}{\partial t}\right) = 0.$$

This is the condition found by BELTRAMI to be the necessary and sufficient condition that the curves be cut orthogonally by a family of surfaces.

We will consider now the general case given by equation (22). Replace t by $\phi(u, v)$; then the quantity

$$\frac{1}{\sqrt{T_1}}\left[\left(T_2 + T_1\frac{\partial \phi}{\partial u}\right)du + \left(T_3 + T_1\frac{\partial \phi}{\partial v}\right)dv\right]$$

must be an exact differential. Expressing this condition we get the equation

$$(25) \qquad \begin{aligned}\left(T_1\frac{\partial T_1}{\partial v} - 2T_1\frac{\partial T_3}{\partial t} + T_2\frac{\partial T_1}{\partial t}\right)\frac{\partial \phi}{\partial u} + \left(2T_1\frac{\partial T_2}{\partial t} - T_2\frac{\partial T_1}{\partial t} - T_1\frac{\partial T_1}{\partial u}\right)\frac{\partial \phi}{\partial v} \\ + \left(2T_1\frac{\partial T_2}{\partial v} - T_2\frac{\partial T_1}{\partial v} - 2T_1\frac{\partial T_3}{\partial u} + T_3\frac{\partial T_1}{\partial u}\right) = 0.\end{aligned}$$

Hence, given any congruence whatever, a function ϕ can be found which furnishes a solution of the problem. And the orthogonal surfaces are given by the equations

$$x = \xi + \frac{l}{\sqrt{T_1}}\frac{\partial f_1}{\partial t}, \qquad y = \eta + \frac{l}{\sqrt{T_1}}\frac{\partial f_2}{\partial t}, \qquad z = \zeta + \frac{l}{\sqrt{T_1}}\frac{\partial f_3}{\partial t},$$

where l is given by quadrature from (22). Hence we have the theorem:

* FORSYTH, *Treatise on Differential Equations*, p. 251.

Given any congruence whatever, there is an infinity of surfaces which cut the curves in points such that the tangents to the curves at these points form a normal congruence. The family of surfaces is given by the integration of a partial differential equation of the first order and the orthogonal surfaces by a quadrature.

From equation (25) it follows that the necessary and sufficient condition that the surfaces $t = \text{const.}$ furnish a solution of the problem is that the functions T_1, T_2, T_3 satisfy the equation

$$2T_1\frac{\partial T_2}{\partial v} - T_2\frac{\partial T_1}{\partial v} - 2T_1\frac{\partial T_3}{\partial u} + T_3\frac{\partial T_1}{\partial u} = 0. \tag{26}$$

It is evident that to this class belong the curves of intersection of triply-orthogonal surfaces; they correspond to the solution $T_2 = T_3 = 0$.

We proceed now to the second part of the problem. The rectilinear congruence of tangents is given by

$$x = \xi + \rho\frac{\partial f_1}{\partial t}, \qquad y = \eta + \rho\frac{\partial f_2}{\partial t}, \qquad z = \zeta + \rho\frac{\partial f_3}{\partial t}. \tag{27}$$

Comparing these expressions with (8) and recalling (17), we find that the necessary and sufficient condition that there exist a function $\phi(u, v)$ such that the developables of the congruence (27) be given by $u = \text{const.}$ and $v = \text{const.}$ is

$$\begin{vmatrix} \frac{\partial^2 f_1}{\partial t^2} & \frac{\partial^2 f_2}{\partial t^2} & \frac{\partial^2 f_3}{\partial t^2} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \end{vmatrix}\frac{\partial \phi}{\partial u} + \begin{vmatrix} \frac{\partial^2 f_1}{\partial t\,\partial u} & \frac{\partial^2 f_2}{\partial t\,\partial u} & \frac{\partial^2 f_3}{\partial t\,\partial u} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \end{vmatrix} = 0, \tag{28}$$

and

$$\begin{vmatrix} \frac{\partial^2 f_1}{\partial t^2} & \frac{\partial^2 f_2}{\partial t^2} & \frac{\partial^2 f_3}{\partial t^2} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix}\frac{\partial \phi}{\partial v} + \begin{vmatrix} \frac{\partial^2 f_1}{\partial t\,\partial v} & \frac{\partial^2 f_2}{\partial t\,\partial v} & \frac{\partial^2 f_3}{\partial t\,\partial v} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix} = 0. \tag{29}$$

From these expressions for $\partial\phi/\partial u$ and $\partial\phi/\partial v$ it is evident that the condition of integrability is not satisfied by every congruence. Since this condition of integrability involves the three functions f_1, f_2, f_3, two of the latter can be chosen arbitrarily and the determination of the third requires the integration of a partial differential equation of the third order in three variables. When this

condition is satisfied, ϕ is found by a quadrature, and, as it involves an arbitrary constant, there will be an infinity of surfaces furnishing a solution of the problem.

We have remarked that, when the formulæ (1) define a triply orthogonal system of surfaces, a solution of the above problem is given by

$$t = \phi = \text{const.}$$

From (28) and (29) we have that the necessary and sufficient condition that the congruence admits this solution is that f_1, f_2, f_3 are particular solutions of equations of the form

$$\begin{aligned} &\frac{\partial^2\theta}{\partial t\,\partial u} + a_1\frac{\partial\theta}{\partial t} + b_1\frac{\partial\theta}{\partial u} = 0, \\ &\frac{\partial^2\theta}{\partial t\,\partial v} + a_2\frac{\partial\theta}{\partial t} + b_2\frac{\partial\theta}{\partial v} = 0. \end{aligned} \tag{30}$$

Hence *the necessary and sufficient condition that the tangents to the curves defined by* (1) *at their intersections with a surface* $t = \text{const.}$ *form a congruence for which the ruled surfaces* $u = \text{const.}$ *and* $v = \text{const.}$ *are developable is that the parametric curves on the two families of surfaces defined by* (1), *when* u *and* v *respectively are constant, form a conjugate system.**

From (28) and (29) it is seen that for any value whatever of ϕ to satisfy the conditions of the problem f_1, f_2, f_3 must satisfy, in addition to the equations (30), two equations of the form

$$\begin{aligned} &\frac{\partial^2\theta}{\partial t^2} + \alpha_1\frac{\partial\theta}{\partial t} + \beta_1\frac{\partial\theta}{\partial u} = 0, \\ &\frac{\partial^2\theta}{\partial t^2} + \alpha_2\frac{\partial\theta}{\partial t} + \beta_2\frac{\partial\theta}{\partial v} = 0. \end{aligned}$$

This shows that the curves $u = \text{const.}$ on the surfaces $v = \text{const.}$ and the curves $v = \text{const.}$ on the surfaces $u = \text{const.}$ must be asymptotic, that is the curves of intersection of these two families must be asymptotic for both surfaces. This can be satisfied only by these lines being rectilinear, so that the congruence is rectilinear. Hence the only case where ϕ can be chosen at will is the evident case where the triple system is formed of the two families of developables of a normal congruence and the surfaces normal to the latter.

* Two curves upon a surface are said to have conjugate directions at a point, when their tangents are parallel to conjugate diameters of the Dupin indicatrix for the point. A double family of curves forms a conjugate system, when the two curves through any point of the surface have conjugate directions at the point. The necessary and sufficient condition that a parametric system be conjugate is that the point equation be of the form (30).

§ 4. Normal Rectilinear Congruences and Linear Complexes Associated with Congruences of Curves.

We propose now the problem of finding a surface S of the congruence, given by a relation of the form $v = \phi(u)$, such that the tangents to the curves satisfying this relation form a normal congruence. This is equivalent to saying that the curves $u = \text{const.}$ on this surface are geodesics. The linear element of S is

$$ds^2 = \Sigma\left(\frac{\partial f_1}{\partial t}\right)^2 dt^2 + 2\left(\Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial u} + \phi'\Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial v}\right)dt\,dv + \Sigma\left(\frac{\partial f_1}{\partial u} + \phi'\frac{\partial f}{\partial v}\right)^2 du^2,$$

where the prime denotes differentiation. The necessary and sufficient condition that the lines $u = \text{const.}$ be geodesics is *

$$2E\frac{\partial F}{\partial t} - F\frac{\partial E}{\partial t} - E\frac{\partial E}{\partial u} = 0, \tag{31}$$

where, as usual, E, F, G are the coefficients in the expression for the linear element of the surface. When the above expressions are substituted in this equation, we get the condition

$$\phi'(u) = \frac{\Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial u}\cdot\Sigma\frac{\partial f_1}{\partial t}\frac{\partial^2 f_1}{\partial t^2} - \Sigma\left(\frac{\partial f_1}{\partial t}\right)^2\cdot\Sigma\frac{\partial f_1}{\partial u}\frac{\partial^2 f_1}{\partial t^2}}{\Sigma\left(\frac{\partial f_1}{\partial t}\right)^2\left[\Sigma\frac{\partial f_1}{\partial v}\frac{\partial^2 f_1}{\partial t^2} + \Sigma\frac{\partial f_1}{\partial t}\frac{\partial^2 f_1}{\partial t\,\partial v}\right] - \Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial v}\cdot\Sigma\frac{\partial f_1}{\partial t}\frac{\partial^2 f_1}{\partial t^2}}. \tag{32}$$

From this expression it follows that for the existence of a function ϕ satisfying the condition of the problem it is necessary and sufficient that the right-hand member of this equation be a function of u alone or a constant; moreover, when this condition is satisfied, ϕ is given by a quadrature. It is readily seen that this condition is not satisfied in general, so that every congruence does not furnish a solution of this problem.

We note that this condition is satisfied when

$$\frac{\partial^2 f_1}{\partial t^2} = \frac{\partial^2 f_2}{\partial t^2} = \frac{\partial^2 f_3}{\partial t^2} = 0,$$

that is, when

$$x = a_1 + a_2 t, \qquad y = b_1 + b_2 t, \qquad z = c_1 + c_2 t, \tag{33}$$

where $a_1, a_2, \cdots, c_2$, are functions of u and v; this congruence is evidently rectilinear. Let a_2, b_2, c_2 satisfy the condition

$$a_2^2 + b_2^2 + c_2^2 = 1,$$

and consequently be the direction-cosines of the lines. When this fact is noted in (32) we find that $\phi'(u)$ is indeterminate. This is what should be expected; for,

* Bianchi, *Lezioni*, p. 146; German translation, p. 150.

if v be replaced by any function of u in (33) the latter define a ruled surface and consequently the lines $u = \text{const.}$ are geodesics.

Again, we remark that when the congruence is given by a triply-orthogonal system the equation (31) reduces to

$$\frac{\partial E}{\partial u} = 0,$$

so that E is a function of t alone. From this it follows that the parameter t can be so chosen that we have

$$E = \Sigma \left(\frac{\partial f_1}{\partial t} \right)^2 = 1.$$

In this case the square of the linear element on the surfaces $v = \text{const.}$ and $u = \text{const.}$ would take the respective forms

$$ds^2 = dt^2 + \lambda du^2,$$

$$ds^2 = dt^2 + \mu dv^2,$$

where λ denotes a function of t and u, and μ a function of t and v. From this we see that the intersections of the surfaces $u = \text{const.}$, $v = \text{const.}$, that is the curves of the congruence, would be geodesics on both surfaces, which is possible only in case these curves be rectilinear.* Hence the only triply-orthogonal system furnishing a solution of this problem is composed of the developable surfaces of a normal congruence and the orthogonal surfaces.

We have incidentally from the preceding discussion:

A family of surfaces of MONGE† *cannot form part of a triply-orthogonal system unless the other surfaces are the developables of the normals to the former.*

The totality of the tangents to the curves of the congruence (1) is a complex. If the equations of a line in this complex are given in the PLÜCKER form

$$bz - cy = p, \qquad cx - az = q, \qquad ay - bx = r,$$

we have

$$a = \frac{\partial f_1}{\partial t}, \qquad b = \frac{\partial f_2}{\partial t}, \qquad c = \frac{\partial f_3}{\partial t},$$

$$p = f_3 \frac{\partial f_2}{\partial t} - f_2 \frac{\partial f_3}{\partial t}, \qquad q = f_1 \frac{\partial f_3}{\partial t} - f_3 \frac{\partial f_1}{\partial t}, \qquad r = f_2 \frac{\partial f_1}{\partial t} - f_1 \frac{\partial f_2}{\partial t}.$$

The necessary and sufficient condition that this complex be linear is that there exist six constants $a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1$, such that the equation

* BIANCHI, *Lezioni*, p. 206; German translation, p. 214.

† DARBOUX, *Leçons*, vol. 1, p. 103.

$$(a_1 - \beta_1 f_3 + \gamma_1 f_2)\frac{\partial f_1}{\partial t} + (b_1 - \gamma_1 f_1 + \alpha_1 f_3)\frac{\partial f_2}{\partial t} + (c_1 - \alpha_1 f_2 + \beta_1 f_1)\frac{\partial f_3}{\partial t} = 0 \tag{34}$$

is satisfied identically. From this we see that given any six constants and two functions f_2, f_3, the determination of f_1, so that (1) defines a congruence whose tangents form a linear complex, requires the integration of a linear equation.

§5. Triple Congruences.

We have remarked that the equations (1) define three congruences C_t, C_u, C_v, according as t, u and v are considered as the parameters of points on the curve. Since equation (6) is symmetrical with respect to t, u, v, it follows that this equation holds for the congruences C_u and C_v as well as for C_t. Recalling the interpretation of this equation, we have the theorem:

In any triple congruence of curves the focal surface is the same for all three congruences.

Consider a point M of the focal surface and the curves Γ_t, Γ_u, Γ_v through this point. All the surfaces of the congruence C_t through Γ_t are tangent to one another and the focal surface at M; likewise for the surfaces of the congruences C_u and C_v through Γ_u and Γ_v respectively. Hence:

All the surfaces of the three congruences through the respective lines of these congruences meeting the focal surface in the same point are tangent to one another and the focal surface at the point. And these respective lines are all tangent to the focal surface.

The most general triply rectilinear congruence is defined by

$$x = (\alpha_1 uv + \beta_1 u + \gamma_1 v) + (a_1 uv + b_1 u + c_1 v + d_1)t,$$
$$y = (\alpha_2 uv + \beta_2 u + \gamma_2 v) + (a_2 uv + b_2 u + c_2 v + d_2)t,$$
$$z = (\alpha_3 uv + \beta_3 u + \gamma_3 v) + (a_3 uv + b_3 u + c_3 v + d_3)t.$$

We have shown elsewhere* that these lines are the intersections of a triple system of hyperbolic paraboloids and furthermore that they furnish the only example of a triple system of surfaces cutting one another along asymptotic lines. In consequence of the above theorem we have the following:

In a triply rectilinear congruence the three lines through a point on the focal surface lie in a plane—the tangent plane to the focal surface at this point.

Consider now a singular line of C_t. Then for $u = \alpha$, $v = \beta$, where (α, β) determine the singular line, the equation (6) is independent of t. For the con-

* Bulletin of the American Mathematical Society, ser. 2, vol. 7, p. 303.

gruence C_u the point of parameter $u = \alpha$ on every curve of the congruence lying on the surface $v = \beta$ is a focal point and therefore the singular line of C_t lies on one of the focal sheets of C_u. We have then this theorem:

The singular lines of any congruence lie on the focal sheets of two other congruences which form with the first a triple congruence. And when one of these congruences has a singular surface, it is one of the sheets of the focal surfaces of the other congruences.

For example, we consider the triple tetrahedral congruence defined by (9). The singular surfaces of C_u and C_v are given by (14) and (13) respectively, and the focal sheet by (12), (13) and (12), (14), respectively.

From the preceding results it follows that the focal surface must be looked upon as comprising the singular lines and singular surfaces, if the theorem to the effect that the focal surface is the same for all three congruences is to be perfectly general. Hence, if C_t has isolated singular lines which do not lie on the focal sheets, C_u and C_v will have curves for some of the focal nappes.

We have seen that the necessary and sufficient condition that the tangents to the curves of C_t at the points of intersection with a surface $t = \text{const.}$ form a congruence for which the ruled surfaces $u = \text{const.}$ and $v = \text{const.}$ are the developables is that the parametric lines on the two families of surfaces, defined by (1) when u and v respectively are given constant values, form conjugate systems. Similar results are true for C_u and C_v. Moreover, the conditions that C_u and C_t possess this property carry with them the conditions for C_v. In consequence of this we have the theorem:

The necessary and sufficient condition that the tangents to the curves of intersection of any two families of surfaces of a triple system at the points of intersection with a surface of the third family form a rectilinear congruence whose developables cut the surfaces of the third family in the same curves as the first two families of surfaces is that the system be triply conjugate.

It is well known that on the tetrahedral surface defined by

$$x = \alpha(a-u)^m(a-v)^n, \quad y = \beta(b-u)^m(b-v)^n, \quad z = \gamma(c-u)^m(c-v)^n,$$

the curves $u = \text{const.}$, $v = \text{const.}$ form a conjugate system. Hence the triple tetrahedral congruences (9) possess the property referred to in the above theorem.

§ 6. Triple Normal Congruences of Curves.

We have found the conditions (24) which the functions f_1, f_2, f_3 in (1) must satisfy in order that there may exist a family of surfaces cutting the curves of the congruence C_t orthogonally. Similar conditions must be satisfied for the same to be true of C_u and C_v.

Thus, we put

$$U_1 = \Sigma \frac{\partial x}{\partial u}\frac{\partial x}{\partial t}, \qquad U_2 = \Sigma \left(\frac{\partial x}{\partial u}\right)^2, \qquad U_3 = \Sigma \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \tag{35}$$

and

$$V_1 = \Sigma \frac{\partial x}{\partial v}\frac{\partial x}{\partial t}, \qquad V_2 = \Sigma \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}, \qquad V_3 = \Sigma \left(\frac{\partial x}{\partial v}\right)^2. \tag{36}$$

The equations for the determination of $u = \chi(t, v)$ and $v = \psi(t, u)$ giving orthogonal surfaces are

$$U_1 dt + U_2 du + U_3 dv = 0,$$
$$V_1 dt + V_2 du + V_3 dv = 0,$$

and the conditions that the integrals involve an arbitrary constant are

$$U_1\left(\frac{\partial U_2}{\partial v} - \frac{\partial U_3}{\partial u}\right) + U_2\left(\frac{\partial U_3}{\partial t} - \frac{\partial U_1}{\partial v}\right) + U_3\left(\frac{\partial U_1}{\partial u} - \frac{\partial U_2}{\partial t}\right) = 0, \tag{37}$$

$$V_1\left(\frac{\partial V_2}{\partial v} - \frac{\partial V_3}{\partial u}\right) + V_2\left(\frac{\partial V_3}{\partial t} - \frac{\partial V_1}{\partial v}\right) + V_3\left(\frac{\partial V_1}{\partial u} - \frac{\partial V_2}{\partial t}\right) = 0. \tag{38}$$

Comparing (21), (35) and (36), we note that

$$U_1 = T_2, \qquad V_1 = T_3, \qquad V_2 = U_3. \tag{39}$$

We see then that the three equations (24), (37) and (38) must be satisfied simultaneously in order that each of the congruences have a family of surfaces cutting them orthogonally.

The first of these equations is satisfied by $T_2 = T_3 = 0$, and from (23) we have that for this case the surfaces $t = \text{const.}$ cut the curves of C_t orthogonally. In consequence of (39) we have that equation (37) also is satisfied when U_3 is zero and then the surfaces $u = \text{const.}$ cut the curves C_u orthogonally. When these conditions are satisfied, equation (38) vanishes and the surfaces orthogonal to C_v are $v = \text{const.}$ From this we have that the congruences in this case are given by the intersections of the surfaces of a triply orthogonal system.

Again we note that these equations are satisfied by

$$T_1 = T_2 = T_3 = U_2 = U_3 = V_3.$$

But in this case the linear element of space is a perfect square and hence the surfaces $t = \text{const.}$, $u = \text{const.}$, $v = \text{const.}$ are developables circumscribing an imaginary circle at infinity.*

The equations of conditions are also satisfied, when these six functions satisfy the eight equations:

* DARBOUX, *Leçons*, vol. 1, p. 148, footnote.

$$\frac{\partial T_2}{\partial v}=\frac{\partial T_3}{\partial u}=\frac{\partial U_3}{\partial t}, \qquad \frac{\partial T_3}{\partial t}=\frac{\partial T_1}{\partial v}, \qquad \frac{\partial T_1}{\partial u}=\frac{\partial T_2}{\partial t}, \qquad \frac{\partial U_2}{\partial v}=\frac{\partial U_3}{\partial u},$$

$$\frac{\partial T_2}{\partial u}=\frac{\partial U_2}{\partial t}, \qquad \frac{\partial U_3}{\partial v}=\frac{\partial V_3}{\partial u}, \qquad \frac{\partial V_3}{\partial t}=\frac{\partial T_2}{\partial v}.$$

A solution of these equations is given by

$$T_1=f(t), \quad U_2=\phi(u), \quad V_3=\psi(v), \quad T_2=av+b, \quad T_3=au+c,$$
$$U_3=at+d,$$

where f, ϕ, ψ are arbitrary functions and a, b, c, d are arbitrary constants. The surfaces orthogonal to the congruence C_t are given by (1), when t is replaced by a function of $auv+bu+cv+e$, the form of the function depending upon f; similarly for the surfaces orthogonal to the congruences C_u and C_v.

It is readily found that the above equations are satisfied also by

$$T_2=F_1(u,t), \qquad T_3=F_2(v,t), \qquad U_3=F_3(u,v),$$

$$U_2=\int\frac{\partial F_3}{\partial u}dv+\int\frac{\partial F_1}{\partial u}dt+\psi(u),$$

$$T_1=\int\frac{\partial F_2}{\partial t}dv+\int\frac{\partial F_1}{\partial t}du+\phi(t),$$

$$V_3=\int\frac{\partial F_2}{\partial v}dt+\int\frac{\partial F_3}{\partial v}du+\omega(v),$$

where $F_1, F_2, F_3, \phi, \psi, \omega$, are arbitrary functions. As in the preceding case the orthogonal surfaces are given by quadratures, but the operations are far less simple.

But after a system of functions satisfying the above equations of condition have been found the determination of the corresponding congruence requires the solutions of the equations

$$\Sigma\left(\frac{\partial x}{\partial t}\right)^2=T_1, \qquad \Sigma\left(\frac{\partial x}{\partial u}\right)^2=U_2, \qquad \Sigma\left(\frac{\partial x}{\partial v}\right)^2=V_3,$$

$$\Sigma\frac{\partial x}{\partial t}\frac{\partial x}{\partial u}=T_2, \qquad \Sigma\frac{\partial x}{\partial t}\frac{\partial x}{\partial v}=T_3, \qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}=U_3.$$

This determination is seen to involve as a particular case the problem of finding all surfaces with a given linear element, so that one realizes the difficulties which arise in a general solution of this problem.

§ 7. Triple Linear Complexes.

We have found that when the functions f_1, f_2, f_3 satisfy an equation of the form

$$(40)\quad (a_1-\beta_1 f_3+\gamma_1 f_2)\frac{\partial f_1}{\partial t}+(b_1-\gamma_1 f_1+\alpha_1 f_3)\frac{\partial f_2}{\partial t}+(c_1-\alpha_1 f_2+\beta_1 f_1)\frac{\partial f_3}{\partial t}=0,$$

where $a_1, \cdots, \gamma_1$, are constants, the complex of the tangents to the congruence C_t is linear. In like manner for the complexes of tangents to the curves of C_u and C_v the following equations must be satisfied:

$$(41)\quad \begin{aligned}&(a_2-\beta_2 f_3+\gamma_2 f_2)\frac{\partial f_1}{\partial u}+(b_2-\gamma_2 f_1+\alpha_2 f_3)\frac{\partial f_2}{\partial u}\\ &\qquad +(c_2-\alpha_2 f_2+\beta_2 f_1)\frac{\partial f_3}{\partial u}=0,\end{aligned}$$

and

$$(42)\quad \begin{aligned}&(a_3-\beta_3 f_3+\gamma_3 f_2)\frac{\partial f_1}{\partial v}+(b_3-\gamma_3 f_1+\alpha_3 f_3)\frac{\partial f_2}{\partial v}\\ &\qquad +(c_3-\alpha_3 f_2+\beta_3 f_1)\frac{\partial f_3}{\partial v}=0,\end{aligned}$$

where $a_2, \cdots, \gamma_3$, are constants.

The equations (40) and (41) may be written in the form *

$$(43)\quad \frac{\partial f_1}{\partial t}=P_1 f_1+Q_1, \qquad \frac{\partial f_1}{\partial u}=P_2 f_1+Q_2,$$

where

$$P_1=\frac{\gamma_1\frac{\partial f_2}{\partial t}-\beta_1\frac{\partial f_3}{\partial t}}{a_1-\beta_1 f_3+\gamma_1 f_2}, \qquad Q_1=-\frac{(b_1+\alpha_1 f_3)\frac{\partial f_2}{\partial t}+(c_1-\alpha_1 f_2)\frac{\partial f_3}{\partial t}}{a_1-\beta_1 f_3+\gamma_1 f_2},$$

and similarly for P_2 and Q_2. Expressing the condition of integrability of (43) and making use of the same in the reduction, we have

$$\left(\frac{\partial P_1}{\partial u}-\frac{\partial P_2}{\partial t}\right)f_1+\left(P_1 Q_2-P_2 Q_1+\frac{\partial Q_1}{\partial u}-\frac{\partial Q_2}{\partial t}\right)=0,$$

or for the sake of brevity,

$$(44)\quad L_1 f_1+M_1=0.$$

Three cases arise for consideration according as f_2 and f_3 do or do not satisfy the two partial differential equations

$$L_1=0, \qquad M_1=0.$$

* The cases where any of the coefficients of the differential quotients in the above equation vanish are of no interest and will be excluded.

If both of these equations are satisfied, we have simply to determine f_1 from (43) by quadratures and then the tangents to the congruences C_t and C_u form linear complexes.

If $M_1 = 0$, but $L_1 \neq 0$, the complexes are not linear for any expression of f_1; also for the case $L_1 = 0$, $M \neq 0$.

Finally, if both L_1 and M_1 are not zero, the necessary and sufficient condition that the two complexes be linear is that f_2 and f_3 satisfy the partial differential equations of the third order

$$\frac{\partial}{\partial t}\frac{M_1}{L_1} = \frac{P_1 M_1}{L_1} - Q_1, \qquad \frac{\partial}{\partial u}\frac{M_1}{L_1} = \frac{P_2 M_1}{L_1} - Q_2;$$

and in this case we take $-M_1/L_1$ for f_1.

In order that all three complexes be linear we must have (43) and

$$\frac{\partial f_1}{\partial v} = P_3 f_1 + Q_3,$$

where the forms of P_3 and Q_3 are readily found. Now the conditions of integrability are (44) and two other equations of similar form which we shall write

$$L_2 f_1 + M_2 = 0, \qquad L_3 f_1 + M_3 = 0.$$

Considerations similar to the preceding show that for all three complexes to be linear the functions f_2 and f_3 must satisfy six partial differential equations. Since the above results hold alike for f_1, f_2 and f_3, we have the theorem:

Associated with every triple congruence of curves there are three complexes of straight lines formed by the tangents to these curves. For one of these complexes to be linear two of the functions f_1, f_2, f_3 may be chosen arbitrarily and the third is found by quadratures; for two of them to be linear it is necessary and sufficient that two of the functions satisfy partial differential equations of the second or third orders and the third function is determined by quadratures or directly; and, finally, for all three to be linear two of the functions must satisfy six partial differential equations of the second or third orders and the third follows as above.

PRINCETON, *January*, 1903.

SIMILAR CONICS THROUGH THREE POINTS*

BY

T. J. I'A. BROMWICH

In a recent number of the Transactions (vol. 4, 1903, p. 103), Professor R. E. ALLARDICE has determined the envelope of the principal axes of a set of similar conics through three points. It may be of interest to point out that the final equation found by him (p. 105, near the foot) can be put in a slightly different form (see equation (8) below), which has the advantage of showing at a glance that the line infinity is a double tangent at the circular points; it is easy to pass directly from that equation to the form here proposed, by algebraic manipulations, but an alternative investigation, *ab initio*, will be given.

Take the center of the circumcircle of the three given points as origin, and let its radius be taken as unit: then if x, y are conjugate complex coördinates, the circumcircle is

$$xy = 1, \qquad \text{or} \qquad |x| = 1. \tag{1}$$

Let the three given points be $x = a, b, c$; and let any particular conic of the set be

$$(x - ty - p)^2 + \lambda(x + ty - q)^2 = r \tag{2}$$

the axes being $x - ty - p = 0$, $x + ty - q = 0$, where $|t| = 1$. Then the condition of similarity is simply that λ is constant; in ALLARDICE's notation,† $(s-1)/(s+1)$ is equal to λ or $1/\lambda$.

The conic (2) cuts the circle (1) in four points, of which three are $x = a, b, c$; let the fourth be $x = d$. Then, if, following MORLEY,‡ we write

$$s_1 = a + b + c, \qquad s_2 = bc + ca + ab, \qquad s_3 = abc \tag{3}$$

it will be found that a, b, c, d are the roots of

$$(x^2 - px - t)^2 + \lambda(x^2 - qx + t)^2 - rx^2 = 0 \tag{4}$$

and so

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† Hereafter to be indicated by A.

‡ Compare, e. g., these Transactions, vol. 1 (1900), p. 100; vol. 4 (1903), p. 1.

$$(5)\qquad \begin{cases} (1+\lambda)(s_1+d)=2(p+\lambda q), \\ (1+\lambda)(s_3+s_2 d)=2t(-p+\lambda q), \\ s_3 d=t^2. \end{cases}$$

Hence we find

$$(6)\quad \begin{cases} 4pt=(1+\lambda)(s_1 t+t^3/s_3-s_3-s_2 t^2/s_3)=(1+\lambda)(t-bc)(t-ca)(t-ab)/s_3, \\ 4\lambda qt=(1+\lambda)(s_1 t+t^3/s_3+s_3+s_2 t^2/s_3)=(1+\lambda)(t+bc)(t+ca)(t+ab)/s_3, \end{cases}$$

and so the two axes are

$$(7)\quad \begin{cases} x-ty=\frac{1}{4}(1+\lambda)(s_1+t^2/s_3-s_3/t-s_2 t/s_3) \\ \qquad\qquad =\frac{1}{4}(1+\lambda)(t-bc)(t-ca)(t-ab)/s_3 t, \\ x+ty=\frac{1}{4}(1+1/\lambda)(s_1+t^2/s_3+s_3/t+s_2 t/s_3) \\ \qquad\qquad =\frac{1}{4}(1+1/\lambda)(t+bc)(t+ca)(t+ab)/s_3 t. \end{cases}$$

From equations (7) it is clear that the envelope of one axis is found from that of the other by changing λ to $1/\lambda$ (or, in A., by changing the sign of s).

Taking the first of equations (7), we see at once that the envelope is a deltoid (3-cusp hypocycloid) with its center at the point $x=\frac{1}{4}(1+\lambda)s_1$; that the cuspidal tangents are given by $t^3=s_3^2$, and the tangents at the vertices by $t^3=-s_3^2$; and that the radius of the vertex-circle is $\frac{1}{4}|1+\lambda|$, while the radius of the cusp-circle is $\frac{3}{4}|1+\lambda|$. Thus the deltoid is completely specified.

The three tangents from the circumcenter $(x=0, y=0)$ are given by $t=bc, ca, ab$, i. e., the tangents are $x-bcy=0$, etc.; and so these three tangents are perpendicular to the sides of the triangle (A., p. 106).

To transform the first of equations (7) to ALLARDICE's form, we observe that if X, Y, Z are trilinears, then

$$X=\tfrac{1}{2}\,[x+bcy-(b+c)]/\sqrt{bc}, \text{ etc.}$$

and so $uX+vY+wZ=0$ gives

$$\left(\frac{u}{\sqrt{bc}}+\frac{v}{\sqrt{ca}}+\frac{w}{\sqrt{ab}}\right)x+(u\sqrt{bc}+v\sqrt{ca}+w\sqrt{ab})y$$

$$-\left[u\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+v\left(\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{c}}\right)+w\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)\right]=0,$$

or say

$$Px+Qy-R=0.$$

If we compare this with the first equation of (7), we find the line-equation of

the envelope

$$PQR = \tfrac{1}{4}(1+\lambda)(Q+Pbc)(Q+Pca)(Q+Pab)/abc.$$

Now*

$$Q + Pbc = 2\sqrt{bc}(u - v\cos C - w\cos B) \text{ etc.}$$

$$PQ = u^2 + v^2 + w^2 - 2vw\cos A - 2wu\cos B - 2uv\cos C$$

$$R = -2(u\cos A + v\cos B + w\cos C),$$

thus the trilinear line-equation is

$$(8)\quad 0 =$$

$$(u\cos A+v\cos B+w\cos C)(u^2+v^2+w^2-2vw\cos A-2wu\cos B-2uv\cos C)$$

$$+(1+\lambda)(u-v\cos C-w\cos B)(v-w\cos A-u\cos C)(w-u\cos B-v\cos A).$$

Since $\lambda = (s-1)/(s+1)$, we have $\lambda + 1 = 2s/(s+1)$. Now compare (8) with the final equation (A., p. 105), and we see that the first term of (8) must be equal to

$$\Sigma[u(u - v\cos C - w\cos B)\{u\cos(B-C) - v\cos B - w\cos C\}]$$

$$-2(u - v\cos C - w\cos B)(v - w\cos A - u\cos C)(w - u\cos B - v\cos A)$$

an identity which is easily verified, although it can hardly be regarded as obvious *a priori*.

ALLARDICE remarks that in the special case $s = -1$, ($\lambda = 0$ or ∞) the deltoid is degenerate; but it may be of interest to add that in this case the conics of the set are all *parabolas;* and it is the envelope of the *infinite* axis which becomes degenerate; the envelope of the axes (in the ordinary sense) of the parabolas is not degenerate, and has its center at the point $x = \frac{1}{4}s_1$; the radius of its vertex-circle is $\frac{1}{4}$ and that of its cusp-circle is $\frac{3}{4}$; its cuspidal tangents are found as in the general case.

From equation (8) we see that the case $\lambda = -1$ is also degenerate; here the conics are all *circles*, and there is no proper envelope. However, equation (8) gives three points, the circumcenter and the two circular points. In this case (as well as when $s = \infty$, $\lambda = +1$, noticed by ALLARDICE) the two axes give the *same* envelope because $\lambda = 1/\lambda$.

QUEEN'S COLLEGE, GALWAY,
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* It will be found necessary to take $(b+c)/\sqrt{bc}$ equal to $-2\cos A$ (not $+2\cos A$), etc., in order to satisfy the conditions $A + B + C = \pi$, $\sqrt{bc}\cdot\sqrt{ca}\cdot\sqrt{ab} = +abc$.

ADDITION (*August* 12, 1903). If the conics are parabolas the following construction gives the vertex-circle very neatly: Bisect the sides of the given triangle in L, M, N; bisect the sides of LMN in P, Q, R; then the circumcircle of PQR is the vertex-circle. For L is given by $x = \frac{1}{2}(b + c)$, M by $x = \frac{1}{2}(c + a)$; so R is given by $x = \frac{1}{4}(c + s_1)$, which is on the vertex-circle $|x - \frac{1}{4}s_1| = \frac{1}{4}$.

This construction is due to Professor H. C. MCWEENEY (University College, Dublin); it can be extended to the general case, but the method is not so simple.